CORRECTIONS

to be made to the article by S. Kleiman and R. Piene

ENUMERATING SINGULAR CURVES ON SURFACES

appearing in "Algebraic geometry — Hirzebruch 70,"

Cont. Math. **241** (1999), 209–238

p. 217b. — In the displayed formula for $cod(\mathbf{D})$, replace 'm_D' by 'm_V'.

p. 219. — In Table 2-1, the value of r for $X_{1,2}$ should not be 1, but 4.

p. 221b. — The proof of Proposition (3.2) should have used Gotzmann's regularity theorem in much the same way that it is used in the proof of Proposition (3.5). So replace the second paragraph in the first proof by the following two.

There exists a map φ from the germ of Y at y into B carrying y to the origin. Since $m \geq \mu - 1$ by hypothesis, φ is smooth, as we'll now show. It suffices to show the surjectivity of the map of tangent spaces, which is equal to the natural map,

$$H^0(\mathcal{L})/H^0(\mathcal{O}_S) \to H^0\big(\mathcal{L}/\mathcal{K}_{C,S}\mathcal{L}\big).$$

Since $\mathcal{N}$ is spanned, it suffices to show the surjectivity of the map,

$$H^0(\mathcal{M}^{\otimes m}) \to H^0\big(\mathcal{M}^{\otimes m}/\mathcal{K}_{C,S}\mathcal{M}^{\otimes m}\big). \tag{3.2.1}$$

To show the surjectivity of (3.2.1), embed S in a projective space P so that $\mathcal{M} = \mathcal{O}_S(1)$, and let $\mathcal{K}_{C,P} \subset \mathcal{O}_P$ be the ideal such that

$$\mathcal{O}_P/\mathcal{K}_{C,P} = \mathcal{O}_S/\mathcal{K}_{C,S}.$$

Now, since $\mathcal{K}_{C,S} \supset \mathcal{J}_{C,S}$, Proposition (3.1) implies that $\mu \geq \dim H^0(\mathcal{O}_S/\mathcal{K}_{C,S})$. Hence, by Gotzmann's regularity theorem [**13**] (see also [**14**, p. 80]), the ideal $\mathcal{K}_{C,P}$ is μ-regular. So $H^1(\mathcal{K}_{C,P}(m))$ vanishes for $m \geq \mu - 1$. Hence the map

$$H^0(\mathcal{O}_P(m)) \to H^0\big((\mathcal{O}_P/\mathcal{K}_{C,P})(m)\big)$$

is surjective. It factors through (3.2.1), so (3.2.1) is surjective. Thus φ is smooth.

p. 225b. — In the third paragraph of the proof of (3.7), replace the third sentence by the following one.

On the other hand, every fiber of $Z(\mathbf{D}) \to H(\mathbf{D})$ is a projective space, and meets $Z_0(\mathbf{D})$ in a nonempty open subset by (3.5).

p. 226m. — In the second paragraph of Section 4, replace the first clause of the first sentence by the following one.

Let $\pi\colon F \to Y$ be a smooth and projective family of (possibly reducible) surfaces, where Y is equidimensional and Cohen–Macaulay, and ...

p. 226m. — In the displayed sequence of principal parts, the the first term should be twisted by D too:

$$0 \to Sym^{i-1}\,\Omega^1_{F/Y}(D) \to \cdots.$$

p. 230t. — In (4.5), replace the final '=' by '$\leq$', getting

$$\ldots\ a+b+2c \leq r+2. \tag{4.5}$$

p. 230m. — In the paragraph that begins, "The genericity hypothesis also implies," replace the first sentence by the following one.

> The genericity hypothesis also implies that X_2 is reduced, Cohen–Macaulay, and equidimensional of codimension 3 in F.

p. 231. — In the second display, replace $w_1 - e$ by $w_1 + e$, and $w_2 + e^2$ by $w_2 - e^2$. In the third display, replace $w_1 - e$ by $w_1 + e$, and $w_2 + e$ by $w_2 - e^2$. In the next to the last paragraph, replace $w_1 - e$ by $w_1 + e$ and $w_2 + e$ by $w_2 - e^2$, and w_2^3 by $w_2 e$. In the display, remove $[X_i]$, and move the sentence following the display down to after the next display.

p. 237. — References 3, 4, 5, and 8 have appeared. The bibliographic data follows.

3. J. Amer. Math. Soc. **13**(2) (2000), 371–410.
4. Duke Math. J. **99**(2) (1999), 311–28.
5. Surveys in Diff. Geom. **5** (1999), 313-39.
8. London Math Society Lecture Note Series **276**, Cambridge Univ. Press, 2000.

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Enumerating singular curves on surfaces

Steven Kleiman and Ragni Piene

Abstract. We enumerate the singular algebraic curves in a complete linear system on a smooth projective surface. The system must be suitably ample in a rather precise sense. The curves may have up to eight nodes, or a triple point of a given type and up to three nodes. The curves must also pass through appropriately many general points. The number of curves is given by a universal polynomial in four basic Chern numbers.

To justify the enumeration, we make a rudimentary classification of the types of singularities using Enriques diagrams, obtaining results like Arnold's. We show that the curves in question do appear with multiplicity 1 using the versal deformation space, Shustin's codimension formula, and Gotzmann's regularity theorem. Finally, we relate our work to Vainsencher's work with up to seven nodes.

1. Introduction

Consider a complete linear system of dimension n on a smooth irreducible complex projective surface. Among the curves in the system, some have r ordinary nodes and no other singularities; they are, for short, *r-nodal*. How many r-nodal curves pass through $n-r$ general points?

Remarkably, at least for $r \leq 8$, if the system is suitably ample, then the number N_r of r-nodal curves is given by a *polynomial* P_r in the four basic Chern numbers of the surface S and the system $|\mathcal{L}|$. These Chern numbers are

$$d := \mathcal{L}\cdot\mathcal{L},\ k := \mathcal{L}\cdot\mathcal{K}_S,\ s := \mathcal{K}_S\cdot\mathcal{K}_S,\ x := c_2(S).$$

A precise assertion is given in Theorem (1.1) below. It can be extended to count curves with other types of singularities; one such extension is Theorem (1.2). The proofs of these theorems is the subject of this paper.

For example, N_1 is the number of nodal curves in a general pencil, and it is given by the "Zeuthen–Segre formula,"

$$N_1 = 3d + 2k + x.$$

This formula was obtained by Hirzebruch [**16**] as a corollary of a general formula for the index sum of a meromorphic vector field. In our case, the index sum is equal to $N_1 + \mathcal{L}\cdot\mathcal{L}$, since $\mathcal{L}\cdot\mathcal{L}$ is just the number of base points.

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In general, the polynomials P_r may be obtained as follows. Consider the formal identity in t,

$$\textstyle\sum_{r\geq 0} P_r t^r/r! = \exp\bigl(\sum_{q\geq 1} a_q t^q/q!\bigr).$$

Thus $P_0 = 1$, and $P_1 = a_1$, and $P_2 = a_1^2 + a_2$, and $P_3 = a_1^3 + 3a_2a_1 + a_3$, and so forth. Set

$$\begin{aligned}
a_1 &:= 3d + 2k + x,\\
a_2 &:= -42d - 39k - 6s - 7x,\\
a_3 &:= 1380d + 1576k + 376s + 138x,\\
a_4 &:= -72360d - 95670k - 28842s - 3888x,\\
a_5 &:= 5225472d + 7725168k + 2723400s + 84384x,\\
a_6 &:= -481239360d - 778065120k - 308078520s + 7918560x,\\
a_7 &:= 53917151040d + 93895251840k + 40747613760s - 2465471520x,\\
a_8 &:= -7118400139200d - 13206119880240k - 6179605765200s + 516524964480x.
\end{aligned}$$

View P_r as a polynomial in d, k, s, x. Then we have the following theorem.

Theorem (1.1) *Assume $\mathcal{L} = \mathcal{M}^{\otimes m} \otimes \mathcal{N}$ where $\mathcal{N}$ is spanned and $\mathcal{M}$ is very ample. If $r \leq 8$ and $m \geq 3r$, then*

$$N_r = P_r(d, k, s, x)/r!\,.$$

Theorem (1.1) is extended in Theorem (1.2) below to enumerate curves with a triple point of a given type and up to three nodes. Specifically, let

$$N(3),\ N(3,2),\ N(3,2,2),\ \text{and}\ N(3,2,2,2)$$

be the numbers of curves having an ordinary triple point and zero to three ordinary nodes, and passing through $n-r$ general points where r is 4, 5, 6, and 7 respectively. In addition, let

$$N(3(2)),\ N(3(2),2),\ \text{and}\ N(3(2)')$$

be the numbers of curves, respectively, (1) having a D_6 singularity, that is, a triple point composed of a tacnode cut transversely by a third branch, (2) having such a triple point and a distant ordinary node, and (3) having an E_7 singularity, that is, a triple point composed of a cusp cut tangentially by a second branch. These curves are also required to pass through $n - r$ general points where r is 6, 7, and again 7, respectively.

Theorem (1.2) *As in Theorem* (1.1)*, assume $\mathcal{L} = \mathcal{M}^{\otimes m} \otimes \mathcal{N}$ where $\mathcal{N}$ is spanned, $\mathcal{M}$ is very ample, and $m \geq 3r$ where r is the appropriate number, now between* 4 *and* 7 *as specified just above. Then the following seven formulas hold:*

$$\begin{aligned}
N(3) &= 15d + 20k + 5s + 5x;\\
N(3,2) &= 45d^2 + (15s + 90k + 30x - 420)d + 40k^2 + (10s + 30x - 624)k\\
&\quad + (5x - 196)s + 5x^2 - 100x;\\
N(3,2,2) &= \bigl(135d^3 + (135x + 45s + 360k - 3150)d^2 + (300k^2 + (60s + 240x - 6849)k\\
&\quad + (-1476 + 30x)s + 45x^2 - 1755x + 18480)d + 80k^3 + (100x - 3276 + 20s)k^2\\
&\quad + ((-1099 + 20x)s + 40x^2 - 1983x + 29946)k - 30s^2 + (10932 - 457x + 5x^2)s\\
&\quad + 5x^3 - 235x^2 + 3120x\bigr)/2!;
\end{aligned}$$

$$\begin{aligned}N(3,2,2,2) = \big(&405d^4 + (-17010 + 1350k + 135s + 540x)d^3 + (1620k^2 + (270s + 1350x\\&- 48573)k + (135x - 7992)s + 270x^2 - 14985x + 239940)d^2 + (840k^3 + (180s + 1080x\\&- 43074)k^2 + ((-11691 + 180x)s + 450x^2 - 29052x + 559398)k - 270s^2 + (-5013x\\&+ 143184 + 45x^2)s + 60x^3 - 4320x^2 + 113910x - 1135080)d + 160k^4 + (40s + 280x\\&- 12168)k^3 + ((-4242 + 60x)s + 180x^2 - 13038x + 284204)k^2 + (-180s^2 + (115156\\&+ 30x^2 - 3687x)s + 50x^3 - 4287x^2 + 144002x - 1977552)k + (-90x + 5408)s^2\\&+ (5x^3 - 783x^2 + 41282x - 807006)s + 5x^4 - 405x^3 + 12150x^2 - 128700x\big)/3!;\end{aligned}$$

$$N(3(2)) = 28x + 168s + 224d + 406k;$$

$$\begin{aligned}N(3(2),2) = &-546x - 7281s - 8316d - 16008k + 28x^2 + 462xk + 308xd\\&+ 168sx + 336sk + 504sd + 812k^2 + 1666kd + 672d^2;\end{aligned}$$

$$N(3(2)') = 252d + 488k + 217s + 42x.$$

These seven formulas were chosen for a reason: they are explicitly used as correction terms in Vainsencher's determination of N_r for $r \leq 7$, although they are not explicitly given in his paper [**28**]. Vainsencher's treatment inspired ours greatly, but ours is more refined: our formulas are more compact; our notion of "suitably ample" is more precise; and our supplementary condition is to pass through $n - r$ general points, not simply to lie in a "suitable general" subsystem of dimension $n - r$. We also go further: Vainsencher was unsure about N_7; we settle N_7 and treat N_8. The seven formulas above are only implicit in our proof of Theorem (1.1), but in Section 4 we explain how to make them explicit to prove Theorem (1.2). In Section 5, we compare and contrast in more detail the nature of the appearance of the formulas in Vainsencher's work with that in ours.

It is possible, by modifying our treatment, to enumerate the curves having any other equisingularity type and passing through the appropriate number $n-r$ general points, at least if $r \leq 8$. Although we don't do this enumeration here, nevertheless we must classify the types of singularities involved to establish the validity of the formulas in Theorems (1.1) and (1.2). In Section 2, we carry out this classification using Enriques diagrams; the results agree with Arnold's classification [**1**] (see also Siersma's paper [**26**]).

To establish the validity of the formulas, we must consider the curves lying in the system and having a given equisingularity type: we must show that these curves form a reduced subsystem of the appropriate codimension. Göttsche [**12**, 5.2] took a first step in this direction, treating nodes in an ad hoc fashion, and his work helped inspire ours. Notably, it is his idea to take $\mathcal{L}$ to be of the form $\mathcal{M}^{\otimes m} \otimes \mathcal{N}$, and not simply of the form $\mathcal{M}^{\otimes m}$. Moreover, he [**12**, 2.2] suggested that it might be sufficient to take $m \geq Cr$ for some universal constant C; in fact, our theorems assert that it is sufficient to take $C = 3$. In Section 3, we study equisingular systems of curves by using the theory of complete ideals and the theory of the versal deformation space, supplemented by our classification in Section 2 and by Shustin's codimension formula [**24**, Thm.].

We treat the eight formulas of Theorem (1.1), but not the seven formulas of Theorem (1.2), in [**19**]. In Section 4 below, we review that treatment for two reasons. First, we can then explain how to modify it to obtain the seven formulas of Theorem (1.2). Second, we can then explain how to use the results we prove in Sections 2 and 3 to obtain an alternative proof of the validity of all the formulas. In [**19**], with an eye toward any r, we establish validity by proving some general results about Enriques diagrams, and by making a direct study of the Hilbert scheme of

clusters of points. Whereas here, we apply Gotzmann's regularity theorem [**13**]; there, we prove an ad hoc regularity result. Here, we work only in characteristic zero; there, we establish the validity in any characteristic for $r \le 8$ if $\mathcal{L} = \mathcal{M}^{\otimes m} \otimes \mathcal{N}$ where $\mathcal{M}$ and $\mathcal{N}$ are spanned, $\mathcal{M}$ is only ample, and $m \ge m_0$ where

$$m_0 := 3r + g^2 + g + 4 - (s+x)/12 \text{ and } g := 1 + \mathcal{M} \cdot (\mathcal{K}_S + \mathcal{M})/2.$$

However, in positive characteristic p, in a given count on an irreducible surface, all the curves may possibly appear with the same multiplicity p^e where $e \ge 1$.

There appears to be, unfortunately, no compact description of the sum,

$$N(3) + N(3,2)t + N(3,2,2)t^2 + N(3,2,2,2)t^3,$$

like the compact exponential description given above for $\sum_r N_r t^r$. (The polynomial giving N_6 fills half a page in [**28**, p. 514]; that giving N_8 fills a page and a half when expanded.) However, the displayed sum is the degree of a cycle class on a suitable space, and this class has an analogous exponential description. The latter is explained in Section 4. We give some other examples of this phenomenon in [**18**]. We conjecture that this cycle-theoretic exponential description holds for any suitably general *algebraic* system of curves on any *algebraic* family of surfaces.

The exponential description of $\sum_r N_r t^r$ is not found in Vainsencher's paper [**28**]. Rather, it was discovered later by Göttsche [**12**, 2.3]. He proved that, if each N_r is given by some polynomial for $r < r_0$ for some r_0, then necessarily $\sum_{r<r_0} N_r t^r$ is of the form $\exp\big(\sum_{q<r_0} A_q t^q/q!\big)$ where each A_q is some *linear* combination of d, k, s, x. This linearity reflects another remarkable property of the polynomials: they continue to work when S is replaced by a surface with several (disjoint) components. (In the summer of 1997 when Göttsche wrote [**12**], considerable evidence suggested that $r_0 = \infty$. Nine months later, Liu [**20**] offered a symplectic proof, and in February 1999, he claimed [pvt. comm.] to have generalized his result to any equisingularity type.) Our use of the exponential was inspired by Göttsche's discovery, but is logically independent of it.

Göttsche related his A_q to the Fourier developments of certain modular forms. These relations suffice to determine the A_q completely when $\mathcal{K}_S$ vanishes. For a K3 surface and for an Abelian surface, the resulting formulas agree with the formulas proved by Bryan and Leung [**3**], [**4**], [**5**] (see also the authors' paper [**18**]). Furthermore, Göttsche explained how to compute the A_q from recursive formulas, such as those of Caporaso and Harris [**6**] for $\mathbf{P}^2$, of Ran [**22**] for $\mathbf{P}^2$, or of Vakil [**29**] for $\mathbf{P}^1 \times \mathbf{P}^1$. For $q \le 8$, he checked that the resulting A_q are equal to the a_q in Theorem (1.1) above, which were obtained independently (they were also obtained from scratch, except for one unknown in a_8).

In short, in Section 2, we make an abstract combinatorial classification of the Enriques diagrams associated to the curves of interest to us, and tabulate the values of their basic numerical characters. In Section 3, we interpret these characters geometrically, and analyze the locus of curves with a given diagram, or equivalently, a given equisingularity type. In Section 4, we derive the formulas and establish their validity; this material rests in part on [**19**]. Finally, in Section 5, we relate Vainsencher's treatment to ours.

2. Minimal Enriques diagrams

In this section we are going to make a rudimentary combinatorial classification of the "minimal Enriques diagrams" that arise from the curves of special interest to us. The diagrams represent the equisingularity types of the curves (this fact was discovered by Enriques [**11**, IV.I], and proved rigorously by Zariski [**31**], although Zariski did not use the language of diagrams; however, this language was used by Casas [**8**, p. 100]).

Given a reduced curve on a smooth surface (which need not be complete), form the configuration of all infinitely near points on all the branches of the curve through all its singular points. Weight each infinitely near point with its multiplicity on the strict transform of the curve. Also, say that one infinitely near point is "proximate" to a second if the first lies above the second and on its strict transform, that is, on the strict transform of the exceptional divisor of the blowup centered at the second. Equipped with this binary relation of proximity, the weighted configuration has an abstract combinatorial structure, which Enriques [**11**, IV.I, pp. 350–51] encoded in a convenient diagram.

By the theorem of strong embedded resolution, all but finitely many infinitely near points are of multiplicity 1, and are proximate only to their immediate predecessors. So it is common to prune off all the infinite unbroken successions of such points, leaving finitely many points. They are known as the *essential* points of the curve (in the terminology of [**15**, 2.2]). Thus, the essential points form a configuration, whose end points either are of higher multiplicity or are proximate to a remote predecessor. We will call the abstract combinatorial structure of this configuration the "minimal Enriques diagram" of the curve. (In [**8**, 3.9], Casas adds all the successors of the remote end points, although these successors are free points of weight 1, and he calls the resulting combinatorial structure the "Enriques diagram" of the curve.)

We formally define a *minimal Enriques diagram* $\mathbf{D}$ to be a finite weighted forest that is equipped with a binary relation and that is subject to the laws stated below. As usual, a *forest* is a disjoint union of trees. A *tree* is a directed graph, without loops; it has a single initial vertex, or *root,* and every other vertex has a unique immediate predecessor. A final vertex is called a *leaf.*

For convenience, we consider a vertex to be one of its own predecessors and one of its own successors, and we call its other predecessors and successors *proper.* We require the *weight* of a vertex V to be an integer m_V at least 1. We denote the binary relation by '$\succ$' and call it *proximity.* As is customary, if a vertex V is proximate to a remote predecessor, then we call V a *satellite*; otherwise, we call V *free.* Thus a root is free.

Furthermore, $\mathbf{D}$ is subject to the following laws:

(Law of Proximity) *A root is proximate to no vertex. If a vertex is not a root, then it is proximate to its immediate predecessor and to at most one other vertex; the latter must be a remote predecessor. If one vertex is proximate to a second, and if a third lies properly between the two, then it too is proximate to the second.*

(Proximity Inequality) *For each vertex V,*

$$m_V \geq \textstyle\sum_{W \succ V} m_W.$$

(Law of Succession) *A vertex V may have at most m_V immediate successors, of which any number may be free, but at most two may be satellites, and they must be satellites of different vertices.*

(Law of Minimality) *Every leaf of weight* 1 *is a satellite.*

After Enriques, we depict $\mathbf{D}$ graphically as follows. We shape the sequence of edges connecting a maximal succession of free vertices into a smooth curve. We shape the sequence of edges connecting a maximal succession of vertices that are all proximate to the same vertex, T say, into a line segment. Its first vertex, V say, is an immediate successor of T. The edge from T to V joins the segment at an angle. Specific diagrams are given in the figures below.

For convenience, set

$$\mathrm{frs}(\mathbf{D}) := \text{the number of free vertices in } \mathbf{D},$$
$$\mathrm{rts}(\mathbf{D}) := \text{the number of roots in } \mathbf{D}.$$

In terms of the preceding numbers, we define the following characters:

$$\begin{aligned} &\dim(\mathbf{D}) := \mathrm{rts}(\mathbf{D}) + \mathrm{frs}(\mathbf{D}); && \delta(\mathbf{D}) := \textstyle\sum_{V\in\mathbf{D}} \binom{m_V}{2}; \\ &\deg(\mathbf{D}) := \textstyle\sum_{V\in\mathbf{D}} \binom{m_V+1}{2}; && r(\mathbf{D}) := \textstyle\sum_V \big(m_V - \sum_{W\succ V} m_W\big); \\ &\mathrm{cod}(\mathbf{D}) := \deg(\mathbf{D}) - \dim(\mathbf{D}); && \mu(\mathbf{D}) := 2\delta(\mathbf{D}) - r(\mathbf{D}) + \mathrm{rts}(\mathbf{D}). \end{aligned}$$

Each of these characters has a geometric meaning, which will be explained in the next section. Note in passing that $r(\mathbf{D})$ measures the total failure of the proximity inequalities to be equalities.

In the next section, we'll need to know which minimal Enriques diagrams $\mathbf{D}$ have $\mathrm{cod}(\mathbf{D}) \le 9$. It suffices to know the $\mathbf{D}$ having only one root R since the others are disjoint unions of these, and so we'll classify them. We'll find that m_R must be 2, 3, or 4, and while we're at it, we'll identify all the $\mathbf{D}$ where m_R is 2 or 3 and $\mathrm{cod}(\mathbf{D})$ is arbitrary, and all the $\mathbf{D}$ where m_R is 4 and $\mathrm{cod}(\mathbf{D}) \le 10$. Our results are given in the figures below.

The figures are labelled according to Arnold's scheme [**1**]. He classified a large number of planar singularities, or bivariate analytic function germs, and gave normal forms for each topological equivalence class. His results imply ours (in characteristic zero), but it is rather easy and elementary to obtain ours directly, as we do now.

Fix a minimal Enriques diagram $\mathbf{D}$ with one root R. We begin the classification by proving a useful little lemma.

Lemma (2.1) *Let S be a vertex. Let V and W be successors of S. Then $m_R \ge m_S \ge m_V$. If $m_S = m_V$, then W is either a successor of V or a predecessor of V; if the latter, then $m_W = m_V$, and V is not remotely proximate to W. If $m_S = 1$ and V is a leaf, then $m_V = 1$ and V is remotely proximate to some proper predecessor of S.*

Indeed, if $V = R$, then the first two assertions are trivial, and the third is vacuous because its hypotheses imply that R is a leaf of weight 1 in violation of the Law of Minimality. So assume that $V \ne R$, and let U be the immediate predecessor of V. Then by the Law of Proximity and the Proximity Inequality, $m_U \ge m_V$, and if $m_U = m_V$, then V is the only vertex proximate to U. Since U is closer to R than V is, we may assume by induction that the lemma holds with U as V.

Hence, $m_R \geq m_U \geq m_V$. If $S = V$, then the first assertion follows, the second is trivial, and the third holds by the Law of Minimality. So assume that $S \neq V$. Then S is a predecessor of U. Hence, $m_R \geq m_S \geq m_U$ by induction. Therefore, $m_R \geq m_S \geq m_V$.

Suppose $m_S = m_V$. Then $m_S = m_U = m_V$. Hence, by induction, W is either a successor of U or a predecessor of U; if the latter, then $m_W = m_U$. If W is a proper successor of U, then it must be a successor of V; otherwise, U would have two immediate successors, and both would be proximate to U by the Law of Proximity, contrary to what we observed above. So assume that W is a predecessor of U. Then $m_W = m_U = m_V$. Moreover, V is not remotely proximate to W. Otherwise, $W \neq U$ since U is the immediate predecessor of V. Hence U too would be proximate to V by the Law of Proximity. Therefore, the Proximity Inequality would yield

$$m_W \geq m_U + m_V = 2m_W.$$

Finally, assume $m_S = 1$ and V is a leaf. Then $m_V = 1$ since $m_S \geq m_V$. By the Law of Minimality, V is proximate to some remote predecessor, and it cannot be a successor of S by the second assertion. The lemma is now proved.

The last assertion of the lemma implies that $m_R \neq 1$.

Assume $m_R = 2$. Let L be a leaf. By the lemma, $2 = m_R \geq m_L$. If $m_L = 2$, then the lemma implies that $\mathbf{D}$ is the diagram A_{2i-1} described in Fig. 2-1.

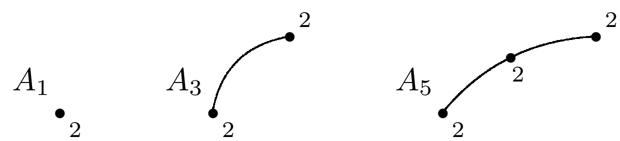

Fig. 2-1. Diagram A_{2i-1} for $i \geq 1$: a succession of i free vertices of weight 2

Suppose $m_L = 1$. Then, by the Law of Minimality, L is proximate to some remote predecessor T. By the lemma, $2 = m_R \geq m_T$. Let U be the immediate predecessor of L. Then, by the Law of Proximity, U is also proximate to T. So, by the Proximity Inequality, $m_T \geq m_U + m_L$. Hence $m_T = 2$ and $m_U = 1$. Let T' be the immediate predecessor of U. Then $T' = T$; otherwise, similarly,

$$2 = m_T \geq m_{T'} + m_U + m_L = 3.$$

The lemma now implies that $\mathbf{D}$ is the diagram A_{2i} described in Fig. 2-2.

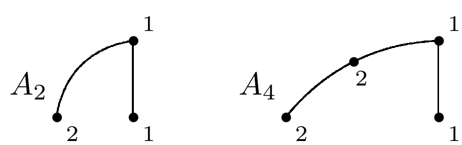

Fig. 2-2. Diagram A_{2i} for $i \geq 1$: a succession of i free vertices of weight 2, followed by two vertices of weight 1 both proximate to the ith vertex

Assume $m_R = 3$. If $\mathbf{D} = \{R\}$, then $\mathbf{D}$ is the diagram D_4, shown in Fig. 2-3.

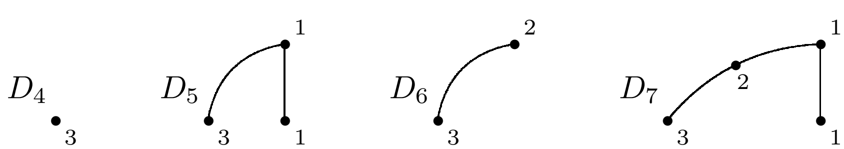

Fig. 2-3. Diagram D_k for $k \geq 4$: a root of weight 3 followed by nothing if $k = 4$, followed by two vertices of weight 1 both proximate to the root if $k = 5$, and followed by the diagram A_{k-5} if $k \geq 6$

Suppose R has an immediate successor S. Then by the lemma $3 = m_R \geq m_S$. First, suppose $m_S = 1$. Let L be a leaf that succeeds S. The lemma implies that $m_L = 1$ and L is remotely proximate to R. Then $L \neq S$ because S is an immediate predecessor of R.

Conceivably, R has a second immediate successor W. If so, then, by the Law of Proximity and the Proximity Inequality, we'd have

$$3 = m_R \geq m_W + m_S + m_L = m_W + 2 \geq 3.$$

So $m_W = 1$. However, with W for S, the argument above would imply that W has a proper successor M that is remotely proximate to R. So, by the Proximity Inequality, we'd have

$$3 = m_R \geq m_W + m_M + m_S + m_L = 4.$$

Thus W does not exist.

If L is an immediate successor of S and the only one, then **D** is the diagram D_5, shown in Fig. 2-3. Suppose not. Then, by the lemma, there is some vertex V strictly between S and L. Then, by the Law of Proximity, V too is proximate to R. If there were a second such vertex W, then by the Proximity Inequality, we'd have

$$3 = m_R \geq m_S + m_V + m_W + m_L = 4.$$

Hence **D** is the diagram E_6, shown in Fig. 2-4.

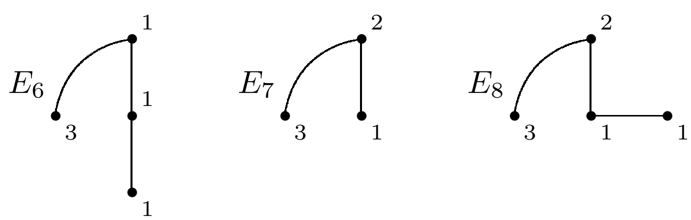

Fig. 2-4. Diagram E_{6l+j} for $l \geq 1$ and $j = 0, 1, 2$: a succession of l vertices of weight 3 followed by three vertices of weight 1 all proximate to the the lth vertex if $j = 0$; followed by two vertices, one of weight 2 and one of weight 1, both proximate to the the lth vertex if $j = 1$; and followed by three vertices, one of weight 2 and then two of weight 1, the first two both proximate to the lth vertex, and the second two both proximate to the $(l+1)$th vertex if $j = 2$

Second, suppose $m_S = 2$. Suppose there is no other vertex proximate to R. Then removing R from **D** produces a new minimal Enriques diagram. It has S as its only root. So it is the diagram A_l for a suitable l by the case $m_R = 2$ above. Hence **D** is the diagram D_{l+5} described in Fig. 2-3.

Suppose there is, other than S, another vertex T proximate to R. Then, by the Proximity Inequality,

$$3 = m_R \geq m_S + m_T = 2 + m_T \geq 3.$$

So $m_T = 1$. Arguing as before, we find that T cannot be a second immediate successor of R; otherwise, T would have a proper successor M that is remotely proximate to R, and so, by the Proximity Inequality,

$$3 = m_R \geq m_T + m_M + m_S = 4.$$

Therefore, T is a successor of S.

Other than S, there is no immediate successor T' of R, because T' would be proximate to R by the Law of Proximity, but would not be a successor of S, contrary to the preceding conclusion.

Other than S and T, no vertex T' is proximate to R; otherwise,

$$3 = m_R \geq m_S + m_T + m_{T'} \geq 4.$$

By the Law of Proximity, any vertex T' between S and T is proximate to R. Hence T is an immediate successor of S. If $\mathbf{D}$ has no further vertices, then it is the diagram E_7 shown in Fig. 2-4.

Conceivably, S has a second immediate successor T'. Arguing as before, suppose so. Then, by the Law of Proximity and the Proximity Inequality,

$$2 = m_S \geq m_T + m_{T'} \geq 2.$$

So $m_{T'} = 1$. However, then T' would have a proper successor M' that is remotely proximate to S. So, by the Proximity Inequality, we'd have

$$2 = m_S \geq m_{T'} + m_{M'} + m_T = 3.$$

Thus T' does not exist.

Suppose $\mathbf{D}$ has a vertex U in addition to R, S and T. Then, by what we've just seen, U must be a successor of T. Let L be a leaf that succeeds U. By the lemma, L is remotely proximate to some proper predecessor S' of U. Then $S' = S$ because S and T are the only vertices proximate to R. Therefore, $U = L$; otherwise, the Proximity Inequality would yield

$$2 = m_S \geq m_T + m_U + m_L = 3.$$

Thus $\mathbf{D}$ is the diagram E_8, shown in Fig. 2-4.

Third and finally, suppose $m_S = 3$. Then there is no other vertex proximate to R because of the Proximity Inequality. Hence removing R from $\mathbf{D}$ produces a new minimal Enriques diagram. It has one less vertex of weight 3, and it has S as its only root. So either it is one of the diagrams already identified, or removing S produces yet another minimal Enriques diagram. Continuing, we conclude that $\mathbf{D}$ is either the diagram E_k, described in Fig. 2-4, or the diagram $J_{l,k}$, described in Fig. 2-5.

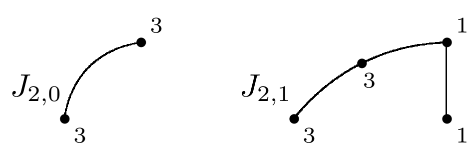

Fig. 2-5. Diagram $J_{l,k}$ for $l \geq 2$ and $k \geq 0$: a succession of $l-1$ vertices of weight 3 followed by the diagram D_{k+4}

Lastly, assume $m_R \geq 4$ and $\operatorname{cod}(\mathbf{D}) \leq 10$. In general, the defining formula for $\operatorname{cod}(\mathbf{D})$ may be rewritten as

$$\operatorname{cod}(\mathbf{D}) = \tbinom{m_R+1}{2} - 2 + \textstyle\sum_{V \in \{\mathbf{D}-R\}} \left(\tbinom{m_V+1}{2} - \tau_V\right)$$

where $\tau_V = 1$ if V is free, and $\tau_V = 0$ if V is a satellite. Hence, $m_R = 4$ since $\operatorname{cod}(\mathbf{D}) \leq 10$. Moreover, $\mathbf{D}$ has no additional free vertex of weight 3 or more. Furthermore, either $\mathbf{D}$ has no additional free vertex of weight 2 or more, and at most two satellites of weight 1, or else $\mathbf{D}$ has one free vertex of weight 2 and no satellites.

A free vertex V of weight 1 does not contribute to $\operatorname{cod}(\mathbf{D})$. However, arguing much as before using the lemma, we find that V is followed by a nonempty succession of vertices of weight 1, all of which are remotely proximate to a predecessor T of V. Necessarily, T is the immediate predecessor of V because V is free, yet proximate to T too. It follows that $\mathbf{D}$ is one of the diagrams shown in Fig. 2-6, Fig. 2-7, and Fig. 2-8.

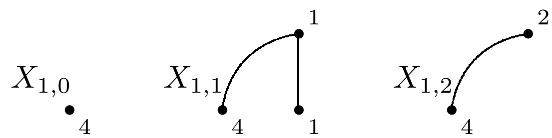

Fig. 2-6. Diagram $X_{1,k}$ for $k = 0, 1, 2$: a root of weight 4 followed by nothing if $k = 0$, followed by two vertices of weight 1 both proximate to the root if $k = 1$, and followed by one vertex of weight 2 if $k = 2$

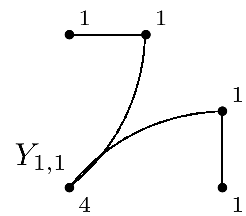

Fig. 2-7. Diagram $Y_{1,1}$: a root of weight 4 followed by two immediate successors, each of weight 1 and each having one successor of weight 1 that is proximate to the root

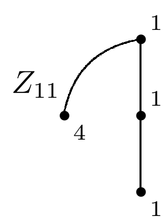

Fig. 2-8. Diagram Z_{11}: a root of weight 4 followed by three vertices of weight 1 all proximate to the root

Our classification is now complete. A simple calculation yields all the corresponding numerical characters, and they are listed in Table 2-1. Finally, here are all the minimal diagrams $\mathbf{D}$ with $\operatorname{cod}(\mathbf{D}) \le 10$ and one root:

$$A_1, \dots, A_{10}, D_4, \dots, D_{10}, J_{2,0}, J_{2,1}, E_6, E_7, E_8, X_{1,0}, X_{1,1}, X_{1,2}, Z_{11}, Y_{1,1}.$$

It is interesting to see the normal forms of the equations of curves with these as associated diagrams. So although we won't need them, we've taken them from Arnold's paper [**1**] and listed them in Table 2-2.

Note in passing that we have not used the Law of Succession. The first time that it is needed is to rule out the following possible $\mathbf{D}$ with $\operatorname{cod}(\mathbf{D}) = 12$: a root of weight 4, followed by a vertex of weight 2, followed by two immediate successors, each being of weight 1 and proximate to the root.

Table 2-1
Characters of Enriques diagrams

type	cod	deg	dim	r	δ	μ
A_{2i-1}	$2i-1$	$3i$	$i+1$	2	i	$2i-1$
A_{2i}	$2i$	$3i+2$	$i+2$	1	i	$2i$
D_{2i}	$2i$	$3i$	i	3	$i+1$	$2i$
D_{2i+1}	$2i+1$	$3i+2$	$i+1$	2	$i+1$	$2i+1$
$J_{l,2i}$	$2i-1+5l$	$3i+6l$	$i+1+l$	3	$i+3l$	$2i-2+6l$
$J_{l,2i+1}$	$2i+5l$	$3i+2+6l$	$i+2+l$	2	$i+3l$	$2i-1+6l$
E_{6l}	$1+5l$	$3+6l$	$l+2$	1	$3l$	$6l$
E_{6l+1}	$2+5l$	$4+6l$	$l+2$	2	$1+3l$	$1+6l$
E_{6l+2}	$3+5l$	$5+6l$	$l+2$	1	$1+3l$	$2+6l$
$X_{1,0}$	8	10	2	4	6	9
$X_{1,1}$	9	12	3	3	6	10
$X_{1,2}$	10	13	3	4	7	11
Z_{11}	10	13	3	2	6	11
$Y_{1,1}$	10	14	4	2	6	11

Table 2-2
Normal forms

type	normal form	type	normal form
A_k	y^2+x^{k+1} $(k \geq 1)$	E_8	x^3+y^5
D_k	x^2y+y^{k-1} $(k \geq 4)$	$X_{1,0}$	$x^4+ax^2y^2+y^4$ $(a^2 \neq 4)$
$J_{2,0}$	$x^3+ax^2y^2+y^6$ $(4a^3+27 \neq 0)$	$X_{1,1}$	$x^4+x^2y^2+ay^5$ $(a \neq 0)$
$J_{2,1}$	$x^3+x^2y^2+ay^7$ $(a \neq 0)$	$X_{1,2}$	$x^4+x^2y^2+ay^6$ $(a \neq 0)$
E_6	x^3+y^4	Z_{11}	$x^3y+y^5+axy^4$
E_7	x^3+xy^3	$Y_{1,1}$	$x^5+ax^2y^2+y^5$ $(a \neq 0)$

3. Geometric interpretation

In the last section, we introduced six numerical characters of an abstract minimal Enriques diagram **D**. In this section, we'll interpret the six geometrically. First, we'll show that three of them represent standard characters of any reduced curve C with **D** as its associated diagram. Then we'll relate the other three to the set of all such C in a given complete linear system $|\mathcal{L}|$.

Fix, once and for all, a nonempty abstract minimal Enriques diagram **D**, a smooth surface S, and an invertible sheaf $\mathcal{L}$ on S. After proving Proposition (3.1), we'll assume that S is irreducible and projective so that the complete linear system $|\mathcal{L}|$ is parameterized by a projective space Y.

Let C be any reduced curve on S. One standard character of C is its total *Milnor number* $\mu(C)$, which can be defined by the formula,

$$\mu(C) := \dim H^0(\mathcal{O}_S/\mathcal{J}_{C,S}),$$

where $\mathcal{J}_{C,S}$ is the Jacobian ideal of C on S; the latter is just the first Fitting ideal of the sheaf of differentials Ω^1_C viewed as an $\mathcal{O}_S$-module.

Define two characters of C using the normalization map $\nu\colon C' \to C$. First, set

$$r(C) := \text{the number of points of } C' \text{ in } \nu^{-1}\operatorname{Supp}(\nu_*\mathcal{O}_{C'}/O_C);$$

in other words, $r(C)$ is the total *number of branches* of C through all its singular points. Second, set

$$\delta(C) := \dim H^0(\nu_*\mathcal{O}_{C'}/O_C);$$

we may call $\delta(C)$ *the genus discrepancy* of C.

These three characters of C are, according to the following proposition, equal to the corresponding characters of $\mathbf{D}$ if $\mathbf{D}$ is the diagram we associate to C in the way described at the beginning of Section 2.

Proposition (3.1) *If C has $\mathbf{D}$ as its associated diagram, then*

$$r(\mathbf{D}) = r(C) \text{ and } \delta(\mathbf{D}) = \delta(C) \text{ and } \mu(\mathbf{D}) = \mu(C).$$

Indeed, consider $r(\mathbf{D})$ and $r(C)$. Both vanish if C is smooth, or equivalently, if $\mathbf{D}$ is empty. To prove that $r(\mathbf{D}) = r(C)$, proceed by induction on the number of vertices in $\mathbf{D}$. Both $r(\mathbf{D})$ and $r(C)$ are "additive" in the set of singular points of C. So we may assume that C has only one singular point, z say. Let R be the corresponding root of $\mathbf{D}$.

Blowup the surface at z. Let E be the exceptional divisor, C' be the strict transform of C, and $z_1, \dots, z_n$ be the points of C' on E. Then the number of branches of C through z is equal to the sum over i of the number of branches of C' through z_i. The latter number is 1 if z_i is a simple point of C'. There are two groups of these points: those whose branch is transverse to E, say $z_1, \dots, z_l$, and those whose branch is not, say $z_{l+1}, \dots, z_k$.

To find l, note that m_R is equal to the intersection number of C' with E, so to the sum, over all the infinitely near points w proximate to z, of the multiplicity at w of the strict transform of C. Each w corresponds to a vertex W in $\mathbf{D}$, proximate to R, unless w is one of $z_1, \dots, z_l$. Hence, we have

$$m_R - \textstyle\sum_{W \succ R} m_W = l.$$

To find $k - l$, let z_i be one of $z_{l+1}, \dots, z_k$. Say z_i corresponds to S_i in $\mathbf{D}$. Consider all the vertices that succeed S_i. Lemma (3.1) implies that they form a single line of succession, say from $V_{i1} := S_i$ to V_{ij_i}; moreover, each V_{ij} is proximate to R as well as to its immediate predecessor, which is $V_{i,j-1}$ if $j \geq 2$; also $m_{V_{ij}} = 1$ for every i, j. Hence, we find

$$\textstyle\sum_{i=l+1}^{k} \sum_{j}^{j_i} \bigl(m_{V_{ij}} - \sum_{W \succ V_{ij}} m_W\bigr) = \sum_{i=l+1}^{k} \bigl(\sum_{j}^{j_i - 1}(1-1) + 1\bigr) = k - l.$$

The remaining points $z_{k+1}, \dots, z_n$ are the singular points of C'. So they correspond to the roots of the minimal Enriques diagram $\mathbf{D}'$ of C'. By induction, the number of branches of C' through $z_{k+1}, \dots, z_n$ is equal to $r(\mathbf{D}')$. Hence

$$r(C) = l + (k - l) + r(\mathbf{D}').$$

Now, $\mathbf{D}'$ is obtained from $\mathbf{D}$ by deleting R and also all the V_{ij}. Therefore, $r(C) = r(\mathbf{D})$.

The equation $\delta(\mathbf{D}) = \delta(C)$ is a version of a famous old result, which is often attributed to Noether and Enriques. For a modern proof, see Théorème (2.11)(ii) on p. 19 of [**9**].

The equation $\mu(\mathbf{D}) = \mu(C)$ follows immediately from the other two, together with the celebrated Milnor–Jung formula,

$$\mu(C) = 2\delta(C) - r(C) + s(C),$$

where $s(C)$ denotes the number of (distinct) singular points of C, which is equal to the number of roots $\operatorname{rts}(\mathbf{D})$. For a proof of the formula, see Corollary 6.4.3 on p. 205 in [**8**]. For more references, see Lipman's review of J.-J. Risler's proof in Math Reviews [MR 46-5334]. The proposition is now proved.

From now on, assume that S is irreducible and projective. Denote by Y the projective space parameterizing the complete linear system $|\mathcal{L}|$, and by $Y(\mathbf{D})$ the subset of points representing curves C with $\mathbf{D}$ as associated diagram. The following proposition interprets the character $\operatorname{cod}(\mathbf{D})$ in terms of the geometry of $Y(\mathbf{D})$ when $|\mathcal{L}|$ is the kind of linear system involved in Theorems (1.1) and (1.2).

Proposition (3.2) *Set $\mu := \mu(\mathbf{D})$. Assume that $\mathcal{L} = \mathcal{M}^{\otimes m} \otimes \mathcal{N}$ where $\mathcal{N}$ is spanned, $\mathcal{M}$ is very ample, and $m \geq \mu - 1$. If $Y(\mathbf{D})$ is nonempty, then it is locally closed in Y, and is smooth and equidimensional. Furthermore,*

$$\operatorname{cod}(Y(\mathbf{D}), Y) = \operatorname{cod}(\mathbf{D}).$$

Indeed, let y be an arbitrary point of $Y(\mathbf{D})$, and C the corresponding curve. Consider the multigerm of C along its singular locus, and a corresponding miniversal deformation base space B. (The formal analytic theory of B was initiated by Schlessinger [**23**], and was algebraized by Artin [**2**] and Elkik [**10**]. An alternative complex analytic theory was developed by Mather, Grauert, and others around the same time, and is explained in [**27**, §4].) The space B is smooth, and its tangent space at the origin is equal to

$$H^0(\mathcal{O}_S/\mathcal{K}_{C,S}) \text{ where } \mathcal{K}_{C,S} := \mathcal{J}_{C,S} + \mathcal{I}_{C,S},$$

where $\mathcal{I}_{C,S}$ is the ideal of C on S.

There exists a map φ from the germ of Y at y into B carrying y to the origin. Since $m \geq \mu - 1$ by hypothesis, φ is smooth, as we'll now show. It suffices to show the surjectivity of the map of tangent spaces, which is equal to the natural map,

$$H^0(\mathcal{L})/H^0(\mathcal{O}_S) \to H^0\big(\mathcal{L}/\mathcal{K}_{C,S}\mathcal{L}\big).$$

Since $\mathcal{N}$ is spanned, it suffices to show the surjectivity of the map,

$$H^0(\mathcal{M}^{\otimes m}) \to H^0\big(\mathcal{M}^{\otimes m}/\mathcal{K}_{C,S}\mathcal{M}^{\otimes m}\big). \tag{3.2.1}$$

To show the surjectivity of (3.2.1), embed S in a projective space P so that $\mathcal{M} = \mathcal{O}_S(1)$, and let $\mathcal{K}_{C,P} \subset \mathcal{O}_P$ be the ideal such that

$$\mathcal{O}_P/\mathcal{K}_{C,P} = \mathcal{O}_S/\mathcal{K}_{C,S}.$$

Now, since $\mathcal{K}_{C,S} \supset \mathcal{J}_{C,S}$, Proposition (3.1) implies that $\mu \geq \dim H^0(\mathcal{O}_S/\mathcal{K}_{C,S})$. Hence, by Gotzmann's regularity theorem [**13**] (see also [**14**, p. 80]), the ideal $\mathcal{K}_{C,P}$ is μ-regular. So $H^1(\mathcal{K}_{C,P}(m))$ vanishes for $m \geq \mu - 1$. Hence the map

$$H^0(\mathcal{O}_P(m)) \to H^0\big((\mathcal{O}_P/\mathcal{K}_{C,P})(m)\big)$$

is surjective. It factors through (3.2.1), so (3.2.1) is surjective. Thus φ is smooth.

The map φ extends to a smooth map $U \to B$ where U is a complex analytic (or an étale) neighborhood of y in Y. The set $U \cap Y(\mathbf{D})$ is carried onto the subset

B^{es} of B whose points represent the equisingular deformations of the multigerm of C. (The formal analytic theory of B^{es} was initiated by Wahl [**30**], and can be algebraized using the work of Artin and Elkik cited above. An alternative complex analytic theory was developed by Teissier around the same time, and is explained in [**27**, §5]; this theory depends in part on Wahl's, but not on Artin's and Elkik's.) The subset B^{es} is closed in B, and is smooth. Hence $Y(\mathbf{D})$ is locally closed in Y, and is smooth. Furthermore,

$$\operatorname{cod}(Y(\mathbf{D}), Y) = \operatorname{cod}(B^{\mathrm{es}}, B).$$

Before proceeding with the proof of Proposition (3.2), note that, together, the equation above and that in (3.2) yield the following corollary, which gives another interpretation of $\operatorname{cod}(\mathbf{D})$.

Corollary (3.3) *Under the conditions of Proposition* (3.2)*, the following formula gives the codimension of the subspace B^{es} of equisingular deformations in a miniversal deformation space B of the multigerm of any curve C in $Y(\mathbf{D})$:*

$$\operatorname{cod}(B^{\mathrm{es}}, B) = \operatorname{cod}(\mathbf{D}).$$

Returning to the proof of Proposition (3.2), recall that the analytic multigerm of C can be realized as the multigerm of a plane curve C' of any suitably large degree m'. (Many authors have written about this assertion; for example, see [**25**] and its references.) Let Y' be the projective space parameterizing all the plane curves of degree m', and let $y' \in Y'$ represent C'. Shustin [**24**, Thm.] proved that, locally at y', the subset $Y'(\mathbf{D})$ is locally closed in Y', and is smooth; moreover, he gave a combinatorial expression for $\operatorname{cod}(Y'(\mathbf{D}), Y')$. Applied to Y', y' in place of Y, y, the discussion before the corollary yields an alternative proof that B^{es} is closed in B, and is smooth. (Shustin [**24**, Rmk. 2] said that this fact about B^{es} "easily follows" from his theorem; doubtless, he envisioned a proof much like ours.) It remains to see that Shustin's combinatorial expression yields $\operatorname{cod}(\mathbf{D})$.

Shustin works, in fact, with plane curve germs, not multigerms; however, the general case is an immediate consequence since the B, resp. B^{es}, of a multigerm is just the product of those of its component germs. Given a germ with central point z, Shustin [**24**, Def. 3, p. 31] defines an integer $c(z)$ as follows. If z is simple, he sets $c(z) := -2$. Otherwise, he denotes the multiplicity of z by m. He blows up the ambient plane at z, forms the exceptional divisor E (which he denotes by Π), and lets $z_1, \ldots, z_n$ be the points on E of the strict transform of the curve germ. For each i, he lets $d(z_i, E)$ be the number of points, including z_i, that are infinitely near to z_i and common to E and the strict transform. Finally, he sets

$$c(z) := \textstyle\sum_{i=1}^{n} \bigl(c(z_i) + d(z_i, E)\bigr) + \binom{m+1}{2} + n - 3.$$

Shustin [**24**, Thm.] proved that $\operatorname{cod}(Y'(\mathbf{D}), Y') = c(z) + 1$. So we must prove that $c(z) + 1 = \operatorname{cod}(\mathbf{D})$, assuming z is multiple. Rewrite the defining equation of $c(z)$ as follows:

$$c(z) + 1 = \textstyle\sum_{i=1}^{n} \bigl(c(z_i) + 1 + d(z_i, E)\bigr) + \binom{m+1}{2} - 2.$$

As in the proof of Proposition (3.1), say that $z_1, \ldots, z_k$ are all the simple points of the strict transform; furthermore, say that at $z_1, \ldots, z_l$ the branch is transverse to E, and at $z_{l+1}, \ldots, z_k$ the branch is not transverse. Finally, for $k+1 \le i \le n$, let $\mathbf{D}_i$ denote the diagram associated to the germ at z_i of the strict transform.

If $1 \le i \le l$, then $c(z_i) = -2$ and $d(z_i, E) = 1$. Hence

$$c(z_i) + 1 + d(z_i, E) = 0 \text{ if } 1 \le i \le l.$$

If z is an ordinary multiple point, then $l = n$; hence,

$$c(z) + 1 = \binom{m+1}{2} - 2 = \operatorname{cod}(\mathbf{D}).$$

Proceeding by induction on the number of vertices in $\mathbf{D}$, assume that

$$c(z_i) + 1 = \operatorname{cod}(\mathbf{D}_i) \text{ if } k + 1 \le i \le n.$$

If $l + 1 \le i \le k$, then $c(z_i) = -2$. Hence

$$c(z) + 1 = \sum_{i=l+1}^{k} \big(d(z_i, E) - 1\big) + \sum_{i=k+1}^{n} \big(\operatorname{cod}(\mathbf{D}_i) + d(z_i, E)\big) + \binom{m+1}{2} - 2.$$

Suppose z_i is one of $z_{l+1}, \dots, z_n$. Then z_i corresponds to a vertex S_i in $\mathbf{D}$, and S_i is an immediate successor of the root, R say. Moreover, $d(z_i, E)$ is equal to the number of successors V_{ij} of S_i that are proximate to R, and among the V_{ij}, only S_i is free. If $l + 1 \le i \le k$, then $m_{V_{ij}} = 1$ by Lemma (3.1). If $k + 1 \le i \le n$, then V_{ij} corresponds to a free vertex in $\mathbf{D}_i$, and S_i corresponds to its root. It follows that, in the preceding display, the right hand side is equal to $\operatorname{cod}(\mathbf{D})$. Thus $c(z) + 1 = \operatorname{cod}(\mathbf{D})$, as required. We have now proved (3.2) and (3.3).

Set $d := \deg(\mathbf{D})$. This fifth character provides a sufficent condition for $Y(\mathbf{D})$ to be nonempty; see Proposition (3.5) below. To prove it, our first step is to construct a weighted configuration $\mathcal{C}$ of infinitely near points of S with $\mathbf{D}$ as its associated diagram.

The construction is straightforward. For every root of $\mathbf{D}$, pick a different point of S, and give it the corresponding weight. Blowup S at each of these points. On each exceptional divisor, pick a different point for every immediate successor in $\mathbf{D}$ of the corresponding root, and give this infinitely near point the corresponding weight. Blow up at each of these points, and on each exceptional divisor, pick a different point for every immediate successor. However, this time take care; if the successor is (remotely) proximate to a root, then the corresponding new point must be taken on the strict transform of the exceptional divisor belonging to this root. Weight each new point appropriately, and continue the construction similarly, taking suitable care with the satellites of $\mathbf{D}$, until all the vertices of $\mathbf{D}$ have been handled. This procedure is possible because $\mathbf{D}$ satisfies the Law of Succession.

Next, we pass from the configuration $\mathcal{C}$ to an ideal $\mathcal{I}$ on S, which is *complete*, or integrally closed. In the construction of $\mathcal{C}$, we perform a succession of blowups, each centered at a point corresponding to a vertex of $\mathbf{D}$. In the end, we obtain a smooth surface S^*. Let $\beta\colon S^* \to S$ be the natural map, and for each vertex V of $\mathbf{D}$, let E_V^* be the full inverse image on S^* of the point corresponding to V. Finally, set

$$\mathcal{I} := \beta_* \mathcal{O}\big(-\textstyle\sum_{V \in \mathbf{D}} m_V E_V^*\big).$$

It is not hard to see that $\mathcal{I}$ is complete.

The theory of complete ideals grew out of the study of complete linear systems with base conditions on surfaces; see [**8**, §8.3] for example. The theory contains a number of interesting and nontrivial results, and two of them will be useful to us. The first asserts the formula,

$$\dim H^0(\mathcal{O}_S/\mathcal{I}) = d \text{ where } d := \deg(\mathbf{D}). \tag{3.4}$$

This formula provides an interpretation of the fifth character d. The formula is a modern version of an old formula [**11**, Vol. II, p. 426], and was proved independently, in various more general settings, by Hoskin [**17**, 5.2, p. 85], Deligne [**9**, 2.13, p. 22], and Casas [**7**]. (Some authors attribute it only to Hoskin and Deligne.) It has been reproved and generalized by others.

The second result speaks to the existence, in the complete system $|\mathcal{L}|$, of a curve C with $\mathcal{C}$ as its configuration, so with $\mathbf{D}$ as its diagram. Denote by $Y(\mathcal{C})$ the subset of Y parameterizing these curves, and by $Y(\mathcal{I})$ the linear subspace of Y associated to $H^0(\mathcal{I}\mathcal{L})$. The result is this:

> *If $\mathcal{I}\mathcal{L}$ is generated by its global sections, then $Y(\mathcal{C})$ is a* nonempty *open subset of $Y(\mathcal{I})$.*

This result can be proved as follows. On S^*, the image of $\beta^*(\mathcal{I}\mathcal{L})$ in $\beta^*\mathcal{L}$ is an invertible sheaf, which is generated by $H^0(\mathcal{I}\mathcal{L})$. So the corresponding linear system has no base points. Consider the smooth members that are transverse to the components of the exceptional divisors E_V. By Bertini's theorem, these members form a nonempty open subset of $Y(\mathcal{I})$. On the other hand, it is not hard to see that this subset is equal to $Y(\mathcal{C})$. (The result is often stated in a local form, where S is replaced by the spectrum of a local ring; see [**8**, §7.2] for example.)

Using the preceding result, we can prove the following proposition, which gives a sufficent condition for $Y(\mathbf{D})$ to be nonempty. The proposition also gives another interpretation of d, which can be paraphrased as follows: d is the number of conditions imposed on the members of $|\mathcal{L}|$ by the requirement that they pass through the infinitely near points of the configuration $\mathcal{C}$ with multiplicities at least as great as the corresponding weights.

Proposition (3.5) *Assume that $\mathcal{L} = \mathcal{M}^{\otimes m} \otimes \mathcal{N}$ where $\mathcal{N}$ is spanned, $\mathcal{M}$ is very ample, and $m \geq d$ where $d := \deg(\mathbf{D})$. Then $Y(\mathbf{D})$ is nonempty. In fact, for any configuration $\mathcal{C}$ and ideal $\mathcal{I}$ with diagram $\mathbf{D}$ as above, $Y(\mathcal{C})$ is a* nonempty *open subset of $Y(\mathcal{I})$, and $\operatorname{cod}(Y(\mathcal{I}), Y) = d$.*

Indeed, embed S in a projective space P so that $\mathcal{M} = \mathcal{O}_S(1)$, and denote the preimage of $\mathcal{I}$ in $\mathcal{O}_P$ by $\mathcal{I}_P$. Then $\mathcal{O}_P/\mathcal{I}_P = \mathcal{O}_S/\mathcal{I}$. So $H^0(\mathcal{O}_P/\mathcal{I}_P) = d$ by Equation (3.4). Hence $\mathcal{I}_P$ is m-regular by Gotzmann's regularity theorem [**13**] (see also [**14**], p. 80). So $\mathcal{I}_P(m)$ is generated by $H^0(\mathcal{I}_P(m))$, and $H^1(\mathcal{I}_P(m))$ vanishes.

It follows that $\mathcal{I}\mathcal{M}^{\otimes m}$ is generated by $H^0(\mathcal{I}\mathcal{M}^{\otimes m})$, and that the map,

$$H^0(\mathcal{M}^{\otimes m}) \to H^0(\mathcal{M}^{\otimes m}/\mathcal{I}\mathcal{M}^{\otimes m}),$$

is surjective.

Since $\mathcal{N}$ is spanned, it follows that $\mathcal{I}\mathcal{L}$ is generated by $H^0(\mathcal{I}\mathcal{L})$ and that the sequence,

$$0 \to H^0(\mathcal{I}\mathcal{L}) \to H^0(\mathcal{L}) \to H^0(\mathcal{L}/\mathcal{I}\mathcal{L}) \to 0,$$

is exact, also on the right. This exactness implies that $Y(\mathcal{I})$ has codimension d in Y, and the result displayed just above implies that $Y(\mathcal{C})$ is a nonempty open subset of $Y(\mathcal{I})$. The proof is now complete.

We turn now to the sixth and last character $\dim(\mathbf{D})$. Although $\mathbf{D}$ is fixed, we make a number of choices when we construct $\mathcal{C}$. In fact, we choose a point in S for each root of $\mathbf{D}$, and a point in a certain exceptional divisor for each remaining free vertex. The choices are not completely arbitrary. However, it is intuitively clear

that the number of "degrees of freedom" is just twice the number of roots plus the number of remaining free vertices, or $\dim(\mathbf{D})$.

To formalize the preceding discussion, note that the correspondence, $\mathcal{C} \mapsto \mathcal{I}$, is injective. Indeed, $Y(\mathcal{C})$ is a nonempty open subset of $Y(\mathcal{I})$ for a suitable $\mathcal{L}$ thanks to Proposition (3.5), or thanks simply to the displayed result preceding it. Conversely, every complete ideal arises from some configuration, although its minimal Enriques diagram is not necessarily $\mathbf{D}$; see [**8**, §8.3] for example.

Form the Hilbert scheme Hilb_S^d. In it, there is a point for every ideal $\mathcal{I}$, thanks to Equation (3.4). The various points form a subset, and it is locally closed by virtue of the work of Nobile and Villamayor [**21**, Thm. 2.6] or that of Lossen [pvt. comm.]. Denote it by $H(\mathbf{D})$, and view $H(\mathbf{D})$ as a reduced subscheme. In terms of $H(\mathbf{D})$, we have the following precise interpretation of the sixth and last character of $\mathbf{D}$.

Proposition (3.6) *In* Hilb_S^d*, the subscheme* $H(\mathbf{D})$ *is smooth and equidimensional. Furthermore,* $\dim H(\mathbf{D}) = \dim(\mathbf{D})$*.*

The proposition is intuitively clear from the discussion preceding it. This discussion is developed into a formal proof in [**19**], but the proof is surprisingly long and involved. (However, it also shows that $H(\mathbf{D})$ is irreducible if S is; moreover, it works in arbitrary characteristic.) Alternatively, the proposition can be derived from Proposition (3.2), Proposition (3.5), and the following lemma; we'll prove the lemma, and then derive the proposition.

Lemma (3.7) *Let* $Z(\mathbf{D})$ *in* $Y \times H(\mathbf{D})$ *denote the incidence scheme, and* $Z_0(\mathbf{D})$ *the preimage of* $Y(\mathbf{D})$ *in* $Z(\mathbf{D})$*. Then* $Z_0(\mathbf{D}) \to Y(\mathbf{D})$ *is bijective.*

Assume that $\mathcal{L} = \mathcal{M}^{\otimes m} \otimes \mathcal{N}$ *where* $\mathcal{N}$ *is spanned and* $\mathcal{M}$ *is very ample. Set* $\mu := \mu(\mathbf{D})$ *and* $d := \deg(\mathbf{D})$*. If* $m \geq d$*, then* $Z(\mathbf{D})/H(\mathbf{D})$ *is a bundle of projective spaces. If* $m \geq \max(d, \mu - 1)$*, then* $Z_0(\mathbf{D})$ *is a dense open subset of* $Z(\mathbf{D})$*, and the projection* $Z(\mathbf{D}) \to Y$ *induces an isomorphism* $Z_0(\mathbf{D}) \xrightarrow{\sim} Y(\mathbf{D})$*.*

Indeed, given $y \in Y(\mathbf{D})$, let C be the corresponding curve, $\mathcal{C}$ its configuration, $\mathcal{I}$ the complete ideal of $\mathcal{C}$, and $h \in H(\mathbf{D})$ the point representing $\mathcal{I}$. Then $(y, h) \in Z(\mathbf{D})$. So $Z_0(\mathbf{D}) \to Y(\mathbf{D})$ is surjective. Now, given $(y, h') \in Z(\mathbf{D})$, let $\mathcal{I}'$ be the complete ideal represented by h', and $\mathcal{C}'$ the corresponding configuration. Every point P' in $\mathcal{C}'$ lies on a strict transform C' of C, and at P' the multiplicity of C' is at least the weight of P' in $\mathcal{C}'$. It follows that $\mathcal{I}'$ contains $\mathcal{I}$. However, both these ideals have colength d in $\mathcal{O}_S$. Hence the two ideals are equal. Hence $h' = h$. Thus, $Z_0(\mathbf{D}) \to Y(\mathbf{D})$ is bijective.

By general principles, for any $\mathcal{L}$, there is a coherent sheaf $\mathcal{Q}$ on $H(\mathbf{D})$ such that $Z(\mathbf{D})$ is equal to $\mathbf{P}(\mathcal{Q})$. In our case, if $m \geq d$, then the fibers of $Z(\mathbf{D})/H(\mathbf{D})$ are all of the same dimension by Proposition (3.5). Hence $\mathcal{Q}$ is locally free because, by hypothesis, $H(\mathbf{D})$ is reduced. Thus $Z(\mathbf{D})/H(\mathbf{D})$ is a bundle of projective spaces.

Assume $m \geq \max(d, \mu - 1)$. Then $Z_0(\mathbf{D})$ is locally closed as $Y(\mathbf{D})$ is so by (3.2). On the other hand, every fiber of $Z(\mathbf{D}) \to H(\mathbf{D})$ is a projective space, and meets $Z_0(\mathbf{D})$ in a nonempty open subset by (3.5). Hence $Z_0(\mathbf{D})$ is dense in $Z(\mathbf{D})$. Therefore, since $Z_0(\mathbf{D})$ is locally closed, it is open. Now, the map $Z_0(\mathbf{D}) \to Y(\mathbf{D})$ is birational since it is bijective (and the characteristic is zero). Hence, by Zariski's Main Theorem, $Z_0(\mathbf{D}) \to Y(\mathbf{D})$ is an isomorphism since it is bijective and since $Y(\mathbf{D})$ is smooth by (3.2). Thus the lemma is proved.

To derive Proposition (3.6), apply Lemma (3.7) with $m \geq \max(\mu - 1, d)$ and $\mathcal{L} = \mathcal{M}^{\otimes m}$ where $\mathcal{M}$ is very ample. Then $Z_0(\mathbf{D})$ is smooth and equidimensional since $Y(\mathbf{D})$ is so by (3.2). Moreover, $Z_0(\mathbf{D}) \to H(\mathbf{D})$ is smooth and surjective, since (3.5) and (3.7) imply that $Z(\mathbf{D}) \to H(\mathbf{D})$ is so, since every fiber meets $Z_0(\mathbf{D})$, and since $Z_0(\mathbf{D})$ is open. Hence, $H(\mathbf{D})$ is smooth and equidimensional. Finally, $\dim H(\mathbf{D})$ is equal to the difference between $\dim Z_0(\mathbf{D})$ and the dimension of the fibers of $Z_0(\mathbf{D}) \to H(\mathbf{D})$. Hence the equation $\dim H(\mathbf{D}) = \dim(\mathbf{D})$ follows from the equations in (3.2) and (3.5). Thus (3.6) is proved, and all six characters of $\mathbf{D}$ have been interpreted.

4. Proofs of the theorems

In this section we prove Theorems (1.1) and (1.2). Our proofs use a recursive formula in r for the cycle class representing the r-nodal curves in an algebraic system on an algebraic family of surfaces. Even though our ultimate interest lies in a linear system on a fixed surface, the added generality is a necessity, not a luxury. Indeed, we proceed inductively by passing to a new system on a family of blowups of the initial family of surfaces. The new system is not linear, and the new family is not constant, even when the initial system and family are so.

Let $\pi\colon F \to Y$ be a smooth and projective family of (possibly reducible) surfaces, where Y is equidimensional and Cohen–Macaulay, and let D be a relative effective divisor on F/Y. Denote by $p_j : F \times_Y F \to F$ the jth projection, by $\Delta \subset F \times_Y F$ the diagonal subscheme, and by $\mathcal{I}_\Delta$ its ideal. Then D is defined by a global section s of the invertible sheaf $\mathcal{O}_F(D)$, and s induces a section s_i of the sheaf of relative twisted principal parts,

$$\mathcal{P}^{i-1}_{F/Y}(D) := p_{2*}\big(p_1^*\mathcal{O}_F(D)/\mathcal{I}^i_\Delta\big) \text{ for } i \geq 1.$$

Denote the scheme of zeros of s_i by X_i. So $X_1 = D$. Furthermore, as a set, X_i consists of the points $x \in F$ at which $D_{\pi(x)}$ has multiplicity at least i. As i varies, the X_i form a descending chain of closed subschemes.

The sheaf $\mathcal{P}^{i-1}_{F/Y}(D)$ fits into the exact sequence,

$$0 \to \mathit{Sym}^{i-1}\,\Omega^1_{F/Y}(D) \to \mathcal{P}^{i-1}_{F/Y}(D) \to \mathcal{P}^{i-2}_{F/Y}(D) \to 0,$$

where the first term is the symmetric power of the sheaf of relative differentials. Hence $\mathcal{P}^{i-1}_{F/Y}(D)$ is locally free of rank $\binom{i+1}{2}$ by induction on i. Therefore, every component of X_i has codimension at most $\binom{i+1}{2}$. Furthermore, if every component has exactly this codimension, then the fundamental class $[X_i]$ is equal to the top Chern class of $\mathcal{P}^{i-1}_{F/Y}(D)$, and so $[X_i]$ can be expressed as a polynomial in the following three Chern classes:

$$v := c_1(\mathcal{O}_F(D)) \text{ and } w_j := c_j(\Omega^1_{F/Y}) \text{ for } j = 1, 2. \tag{4.1}$$

For example, in this way we obtain the following three expressions, which we need for the proofs of Theorems (1.1) and (1.2):

$$
\begin{aligned}
[X_2] &= v^3 + w_1v^2 + vw_2;\\
[X_3] &= v^6 + 4w_1v^5 + (5w_1^2 + 5w_2)v^4 + (2w_1^3 + 11w_1w_2)v^3 + (6w_2w_1^2 + 4w_2^2)v^2 + 4vw_1w_2^2;\\
[X_4] &= v^{10} + 10w_1v^9 + (15w_2 + 40w_1^2)v^8 + (82w_1^3 + 111w_1w_2)v^7\\
&\quad + (91w_1^4 + 315w_2w_1^2 + 63w_2^2)v^6 + (52w_1^5 + 429w_2w_1^3 + 324w_1w_2^2)v^5\\
&\quad + (12w_1^6 + 282w_2w_1^4 + 593w_2^2w_1^2 + 85w_2^3)v^4 + (72w_2w_1^5 + 464w_2^2w_1^3 + 259w_1w_2^3)v^3\\
&\quad + (132w_2^2w_1^4 + 246w_2^3w_1^2 + 36w_2^4)v^2 + (72w_2^3w_1^3 + 36w_1w_2^4)v.
\end{aligned}
$$

Let $b\colon F' \to F \times_Y F$ be the blowup along Δ, and set $p_j' := p_j \circ b$. Then $p_2'\colon F' \to F$ is again a smooth and projective family of surfaces; in fact, over a point x of F, the fiber $F_x' := p_2'^{-1}x$ is just the blowup of the fiber $F_{\pi x} := \pi^{-1}\pi x$ at x. Set $F_i := p_2'^{-1}(X_i)$, and let $\pi_i : F_i \to X_i$ be the restriction of p_2' to F_i. In sum, we have the following diagram:

$$
\begin{tikzcd}
F \arrow[d, "\pi"'] & F \times_Y F \arrow[l, "p_1"'] \arrow[d, "p_2"] & F' \arrow[l, "b"'] \arrow[d, "p_2'"'] & F_i \arrow[l, hook'] \arrow[d, "\pi_i"'] \\
Y & F \arrow[l, "\pi"'] \arrow[r, equal] & F & X_i \arrow[l, hook']
\end{tikzcd}
$$

The pullback $p_1^{-1}D$ is defined by p_1^*s. The restriction of p_1^*s to $p_2^{-1}X_i$ is not simply a section of the restriction of $p_1^*\mathcal{O}_F(D)$, but is also a section of the restriction of its subsheaf $\mathcal{I}_\Delta^i(p_1^{-1}D)$, in view of the definition of X_i. Hence the restriction of $p_1'^*s$ to F_i is a section of the restriction of $\mathcal{O}_{F'}(p_1'^{-1}D - ib^{-1}\Delta)$. Therefore, the difference of the restrictions,

$$D_i := (p_1'^{-1}D)|F_i - (ib^{-1}\Delta)|F_i,$$

is a relative effective divisor on F_i/X_i.

Let $\mathbf{D}$ be an abstract minimial Enriques diagram. In Section 3, we associated some loci to it, and made a study of them. We'll now continue that study in the present context, and apply it to prove Theorems (1.1) and (1.2). First, set $d := \deg(\mathbf{D})$, and in the relative Hilbert scheme $\mathrm{Hilb}^d_{F/Y}$, form the subset $H_{F/Y}(\mathbf{D})$ parameterizing the complete ideals sitting on the fibers of F/Y and having $\mathbf{D}$ as diagram. Just as before, $H_{F/Y}(\mathbf{D})$ is locally closed, and we'll view it as a reduced subscheme. Next, in Y form the subset $Y(\mathbf{D})$ of points over which the fiber of D is a reduced curve with $\mathbf{D}$ as diagram. Next, form the scheme

$$Z(\mathbf{D}) := H_{F/Y}(\mathbf{D}) \bigcap \mathrm{Hilb}^d_{D/Y},$$

and form the set-theoretic preimage $Z_0(\mathbf{D})$ of $Y(\mathbf{D})$ in $Z(\mathbf{D})$. Finally, form the set-theoretic image of $Z(\mathbf{D})$ in Y, and the image's closure $U(\mathbf{D})$.

Assume for a moment that $F = S \times Y$ where S is a smooth, irreducible, projective surface. Assume that Y is the parameter projective space of a complete linear system $|\mathcal{L}|$ on S, and that $D \subset F$ is the total space. Then $\mathrm{Hilb}^d_{F/Y}$ is equal to $Y \times \mathrm{Hilb}^d_S$; so $H_{F/Y}(\mathbf{D})$ is equal to $Y \times H(\mathbf{D})$ where $H(\mathbf{D})$ is the locus of Section 3. Furthermore, $Y(\mathbf{D})$, $Z(\mathbf{D})$, and $Z_0(\mathbf{D})$ are equal to their counterparts in Section 3, but $U(\mathbf{D})$ is new. If we assume in addition that $\mathcal{L}$ is suitably ample, then we may say more about these loci, as the following proposition asserts.

Proposition (4.2) *Preserve the preceding conditions. Let $1 \le r \le 8$. Assume that $\mathcal{L} = \mathcal{M}^{\otimes m} \otimes \mathcal{N}$ where $\mathcal{N}$ is spanned, $\mathcal{M}$ is very ample, and $m \ge 3r$. Set $s := \operatorname{cod}(\mathbf{D})$.*

(1) *Assume $s \le r$. Then $Y(\mathbf{D})$ is locally closed, smooth, and equidimensional, and $\operatorname{cod}(Y(\mathbf{D}), Y) = s$. Also, $Z(\mathbf{D})$ is reduced, $Z_0(\mathbf{D})$ is dense and open in $Z(\mathbf{D})$, and the projection induces an isomorphism of reduced schemes, $Z_0(\mathbf{D}) \xrightarrow{\sim} Y(\mathbf{D})$. Finally, $Y(\mathbf{D})$ is open and dense in $U(\mathbf{D})$.*

(2) *Independent of $\mathbf{D}$, there is a closed subset Y' of Y with $\operatorname{cod}(Y', Y) \ge r+1$ such that Y' contains $Y(\mathbf{D})$ if $s \ge r+1$. Moreover, Y' contains every point of Y over which the fiber of D is a curve with a multiple component.*

Indeed, let's first check the following two inequalities when $s \le 10$:

$$\mu(\mathbf{D}) - 1 \le s \text{ and } \deg(\mathbf{D}) \le 3s. \tag{4.3}$$

Both sides of both inequalities are "additive," so we may assume that $\mathbf{D}$ has only one root. Then both inequalities can be checked by inspecting Table 2-1. (In fact, the second inequality holds without any bound on s; see [**19**].)

Combining Display (4.3) with Propositions (3.2) and (3.5) and with Lemma (3.7), we obtain the first two assertions in Part (1). So $Z_0(\mathbf{D})$ and $Z(\mathbf{D})$ have the same closure in $Y \times \operatorname{Hilb}^d_S$. This closure projects onto $U(\mathbf{D})$ since Hilb^d_S is projective. Now, $Z_0(\mathbf{D})$ maps onto $Y(\mathbf{D})$. Hence $Y(\mathbf{D})$ is dense in $U(\mathbf{D})$. Therefore, since $Y(\mathbf{D})$ is locally closed, it is open in $U(\mathbf{D})$.

To prove Part (2), note that there are only finitely many minimal Enriques diagrams $\mathbf{D}'$ with $\deg(\mathbf{D}') \le 3r$. For each such $\mathbf{D}'$, form $Z(\mathbf{D}')$ in $Y \times H(\mathbf{D}')$, and $U(\mathbf{D}')$ in Y. Then $Z(\mathbf{D}')/H(\mathbf{D}')$ is a fiber bundle by Lemma (3.7). The dimensions of its fibers and of its base are given by Propositions (3.5) and (3.6). Hence,

$$\operatorname{cod}(U(\mathbf{D}'), Y) \ge \operatorname{cod}(\mathbf{D}').$$

If $\operatorname{cod}(\mathbf{D}') \le r$, then equality holds, and $Y(\mathbf{D}')$ is open and dense in $U(\mathbf{D}')$ by Part (1) applied with $\mathbf{D}'$ for $\mathbf{D}$.

Let Y'' be the union of the $U(\mathbf{D}')$ with $\operatorname{cod}(\mathbf{D}') \ge r+1$. Let Y''' be the union of the differences $U(\mathbf{D}') - Y(\mathbf{D}')$ with $\operatorname{cod}(\mathbf{D}') = r$. Set $Y' := Y'' \cup Y'''$. Then $\operatorname{cod}(Y', Y) \ge r+1$. To prove that Y' contains $Y(\mathbf{D})$ if $s \ge r+1$, we need to develop a little more of the general theory of diagrams.

Call a minimal Enriques diagram $\mathbf{D}'$ a *subdiagram* of $\mathbf{D}$, and say that $\mathbf{D}$ *contains* $\mathbf{D}'$, if $\mathbf{D}'$ consists of some of the same vertices, equipped with possibly smaller weights and with the induced relations of succession and proximity. For example, A_{2i-1} is a subdiagram of A_{2i} and of A_{2i+1}, but A_{2i} is not a subdiagram of A_{2i+1}. Also, D_{2i} is a subdiagram of D_{2i+1} and of D_{2i+2}, but D_{2i+1} is not a subdiagram of D_{2i+2}. The following lemma asserts the existence of a convenient subdiagram (it is proved in [**19**] for any $r \ge 1$).

Lemma (4.4) *Let $1 \le r \le 8$. If $\operatorname{cod}(\mathbf{D}) \ge r+1$, then $\mathbf{D}$ contains a minimal Enriques subdiagram $\mathbf{D}'$ such that $\operatorname{cod}(\mathbf{D}') \ge r$ and $\deg(\mathbf{D}') \le 3r$.*

Assuming the lemma, let's complete the proof of the proposition. Assume $s \ge r+1$. Let y be an arbitrary point of $Y(\mathbf{D})$, and C the corresponding curve. Use the notation and constructions of Sections 2 and 3. Form the corresponding weighted configuration $\mathcal{C}$ of infinitely near points of S. The lemma provides a certain subdiagram $\mathbf{D}'$ of $\mathbf{D}$. Let $\mathcal{C}'$ be the corresponding subconfiguration of $\mathcal{C}$.

Let $\mathcal{I}$ and $\mathcal{I}'$ be the associated complete ideals on S. Letting m'_V denote the weight of a vertex $V \in \mathbf{D}'$ we have

$$\mathcal{I} := \beta_*\mathcal{O}\big(-\textstyle\sum_{V\in\mathbf{D}} m_V E_V^*\big) \subset \beta_*\mathcal{O}\big(-\textstyle\sum_{V\in\mathbf{D}'} m'_V E_V^*\big) = \mathcal{I}';$$

the latter equality does not hold by definition, but can be derived using the Leray Spectral Sequence.

The inclusion $\mathcal{I} \subset \mathcal{I}'$ induces an inclusion, $H^0(\mathcal{I}\mathcal{L}) \subset H^0(\mathcal{I}'\mathcal{L})$. In turn, the latter yields $Y(\mathcal{I}) \subset Y(\mathcal{I}')$. However, $y \in Y(\mathcal{I})$ and $Y(\mathcal{I}') \subset U(\mathbf{D}')$. So, if $\operatorname{cod}(\mathbf{D}') \geq r+1$, then $y \in Y''$. Suppose $\operatorname{cod}(\mathbf{D}') = r$. Then $\mathbf{D}' \neq \mathbf{D}$ since $s \geq r+1$. Hence $y \notin Y(\mathbf{D}')$. So $y \in Y'''$. Since y is arbitrary, therefore $Y(\mathbf{D}) \subset Y'$.

Finally, let y be a point of Y over which the fiber of D is a curve with a multiple component. On that component, pick r distinct points, and let $\mathcal{I}'$ be the product of the squares of their maximal ideals. Then $\mathcal{I}'$ is a complete ideal with diagram $\mathbf{D}' := rA_1$. Now, $\operatorname{cod}(\mathbf{D}') = r$ and $\deg(\mathbf{D}') = 3r$. Moreover, $y \in Y(\mathcal{I}')$ and $y \notin Y(\mathbf{D}')$. So $y \in Y'''$. Thus Proposition (4.2) is proved, given Lemma (4.4).

To prove Lemma (4.4), note that $\mathbf{D}$ always contains A_1, and recall from Table 2-1 that $\deg(A_1) = 3$. Hence, if $r = 1$, we may take A_1 as $\mathbf{D}'$. So assume $r \geq 2$. Proceed by induction on the number of roots.

First, assume that $\mathbf{D}$ has only one root, say R of weight m. If $m \geq 4$, then we may take $\mathbf{D}'$ to be $X_{1,0}$ if $r \geq 4$ since $\operatorname{cod} X_{1,0} = 8$ and $\deg X_{1,0} = 10$; moreover, then we may take $\mathbf{D}'$ to be D_4 if $r = 3, 2$ since $\operatorname{cod} D_4 = 4$ and $\deg D_4 = 6$. If $m = 2$, then $\mathbf{D}$ is A_k with $k \geq r+1$, and we may take $\mathbf{D}'$ to be A_r if r is odd and to be A_{r+1} if r is even.

Suppose $m = 3$. Then $\mathbf{D}$ is of type D, E, or J. If $\mathbf{D}$ is either $J_{l,j}$ or E_{6l+j} where $l \geq 2$, then $\mathbf{D}$ begins with a succession of two vertices of weight 3. Hence, then we may take $\mathbf{D}'$ to be $J_{2,0}$ if $r \geq 4$ since $\operatorname{cod} J_{2,0} = 10$ and $\deg J_{2,0} = 12$. Moreover, then we may take $\mathbf{D}'$ to be A_3 if $r = 3, 2$ since $\operatorname{cod} A_3 = 3$ and $\deg A_3 = 6$.

Suppose that $\mathbf{D}$ is either E_7 or E_8. Then $r \leq 7$. Also, E_8 contains E_7, and E_7 contains A_3. Hence, we may take $\mathbf{D}'$ to be E_7 if $r \geq 4$ since $\operatorname{cod} E_7 = 7$ and $\deg E_7 = 10$. Moreover, we may take $\mathbf{D}'$ to be A_3 if $r = 3, 2$.

Suppose $\mathbf{D}$ is E_6. Then $r \leq 5$. Also, E_6 contains A_2. Hence, we may take $\mathbf{D}'$ to be E_6 if $r \geq 3$ since $\operatorname{cod} E_6 = 6$ and $\deg E_6 = 9$. Moreover, we may take $\mathbf{D}'$ to be A_2 if $r = 2$ since $\operatorname{cod} A_2 = 2$ and $\deg A_2 = 5$.

Finally, assume that $\mathbf{D}$ has more than one root. Let $\mathbf{D}_1$ be a connected component, and $\mathbf{D}_2$ its complement; so

$$\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2.$$

Set $s_i := \operatorname{cod}(\mathbf{D}_i)$. Now, $\mathbf{D}_1$ has only one root. If $s_1 \geq r+1$, then by the one-root case, already $\mathbf{D}_1$ contains a $\mathbf{D}'$ such that $\operatorname{cod}(\mathbf{D}') \geq r$ and $\deg(\mathbf{D}') \leq 3r$. If $s_1 = r$, then $\deg(\mathbf{D}_1) \leq 3s_1$ by the second inequality of Display (4.3) applied to $\mathbf{D}_1$; so we may take $\mathbf{D}_1$ as $\mathbf{D}'$.

Suppose $s_1 \leq r-1$. Set $r_2 := r - s_1$. Then $s_2 \geq r_2 + 1$ because

$$s_1 + s_2 = \operatorname{cod}(\mathbf{D}_1) + \operatorname{cod}(\mathbf{D}_2) = \operatorname{cod}(\mathbf{D}) \geq r+1,$$

by the definition of s_i, by the additivity of 'cod', and by hypothesis. So, by induction, we may assume that $\mathbf{D}_2$ contains a $\mathbf{D}'_2$ such that $\operatorname{cod}(\mathbf{D}'_2) \geq r_2$ and $\deg(\mathbf{D}'_2) \leq 3r_2$. Set

$$\mathbf{D}' := \mathbf{D}_1 + \mathbf{D}'_2.$$

Then by the additivity of 'cod' and of 'deg' we have

$$\begin{aligned}\operatorname{cod}(\mathbf{D}') &= \operatorname{cod}(\mathbf{D}_1) + \operatorname{cod}(\mathbf{D}_2') \geq s_1 + r_2 = r, \text{ and}\\ \deg(\mathbf{D}') &= \deg(\mathbf{D}_1) + \deg(\mathbf{D}_2') \leq 3s_1 + 3r_2 = 3r,\end{aligned}$$

the last inequalities coming from Display (4.3). The lemma is now proved.

Let $\pi\colon F \to Y$ and D be arbitrary again. Let $1 \leq r \leq 8$. In Proposition (4.2), the hypotheses concerning $\mathcal{L}$ no longer apply, but in both Parts (1) and (2), the assertions still make sense. Make the genericity hypothesis on F/Y and D that these assertions hold.

The genericity hypothesis implies immediately that the class,

$$u(D,s) := [U(sA_1)] \text{ for } 1 \leq s \leq r,$$

is of codimension s on Y, and that it enumerates the s-nodal curves in the family D/Y, since they form a dense subset of $U(sA_1)$. For convenience, extend the definition of $u(D,s)$ by setting

$$u(D,0) := 1, \text{ and } u(D,s) := 0 \text{ for } s \leq -1.$$

Our next step is to express $u(D,r)$ in terms of the three Chern classes v, w_1 and w_2 of Display (4.1). We'll find that $u(D,r)$ can be expressed as a polynomial of degree r in the following classes:

$$y(a,b,c) := \pi_* v^a w_1^b w_2^c \text{ where } a,b,c \geq 0 \text{ and } a+b+2c \leq r+2. \tag{4.5}$$

We'll just sketch the procedure here; more details are found in [**19**].

Consider the closed subset Y' of Y described in Part (2) of Proposition (4.2). Since $\operatorname{cod}(Y',Y) \geq r+1$, we may replace Y by $Y - Y'$ without affecting the validity of our expression for $u(D,r)$. Thus we may (and will) assume that every fiber of D/Y is reduced and its diagram has cod at most r.

The genericity hypothesis also implies that X_2 is reduced, Cohen–Macaulay, and equidimensional of codimension 3 in F. Indeed, there is a natural embedding of F in $\operatorname{Hilb}^3_{F/Y}$, given by sending a point of F to the square of its maximal ideal, and the image of F is simply $H_{F/Y}(A_1)$. Hence, X_2 is isomorphic to $Z(A_1)$, so reduced of codimension 3. Similarly, X_3 is reduced of codimension 6 if $4 \leq r \leq 8$, and X_4 is reduced of codimension 10 if $r = 8$. Of course, X_3 is empty if $r \leq 3$, and X_4 is empty if $r \leq 7$ since the fibers of D/Y are now well behaved.

The genericity hypothesis further implies that the analogous genericity hypothesis holds for F_i/X_i and D_i for $i = 2,3$ provided r is replaced by $r-1$, respectively $r-4$, and this number is at least 1. However, there is one exception: over every point of X_4, the fiber of D_2 contains twice the exceptional divisor, and so is nonreduced. Moreover, we have the following recursive formula:

$$ru(D,r) = \pi_*\big(u(D_2, r-1) - u(D_3, r-4) + 3281u(D_4, r-8)\big). \tag{4.6}$$

All these statements are discussed in detail in [**19**]; here, we'll accept them as true.

Intuitively, the preceding statements make sense. For example, on the left side of Formula (4.6), the r is present because, over a point of $Y(rA_1)$, the fiber of X_2 consists of r distinct points. On the right, the term $u(D_2, r-1)$ enumerates the $(r-1)$-nodal curves C in the family D_2/X_2. If C is transverse to the exceptional curve E in its ambient fiber of F_2/X_2, then C is the proper transform of an r-nodal curve in the family D/Y, the rth node being at the blowup center. If a C were

tangent to E, then it would be the transform of a curve with a cusp and $(r-1)$ nodes; however, the locus of these curves is $Y(A_2+(r-1)A_1)$, so empty since Y' is now empty. Similarly, no C has one node on E and $(r-1)$ nodes off it.

However, a C can have three nodes on E if $4 \le r \le 8$. Then $C = E + C'$ where C' is an $(r-4)$-nodal curve transverse to E. Such a C' is the proper transform of a curve with an ordinary triple point at the blowup center. So C' is an $(r-4)$-nodal curve in the family D_3/X_3. These curves are enumerated by the class $u(D_3, r-4)$. Thus $u(D_3, r-4)$ appears with a minus sign on the right in Formula (4.6).

The multiplier 3281 is more problematical. Certainly, when $r = 8$, some multiple of $u(D_4, r-8)$ will appear in Formula (4.6), but the multiplier might well vary with F/Y and D. In [**19**], residual intersection theory is used to prove that the multiplier does not vary; then its value is determined from a particular case.

To express $u(D,r)$ as a polynomial in the classes $y(a,b,c)$ of Display (4.5), we proceed formally by recursion on r using Formula (4.6) and the formulas for the $[X_i]$ displayed early in the section. First, for $r = 1$, we obtain

$$\begin{aligned} u(D,1) &= \pi_* u(D_2, 0) = \pi_*[X_2] = \pi_*(v^3 + w_1 v^2 + v w_2) \\ &= y(3,0,0) + y(2,1,0) + y(1,0,1). \end{aligned}$$

To proceed, set $e := [b^{-1}\Delta]$, which is the class of the exceptional divisor. To lighten the notation, let e, v, and w_j also denote their own pullbacks. Then

$$c_1(\mathcal{O}_{F_i}(D_i)) = v - ie,\ c_1(\Omega^1_{F'/F}) = w_1 + e,\ \text{and}\ c_2(\Omega^1_{F'/F}) = w_2 - e^2.$$

So, using Formula (4.6) and recursion in r, express $u(D,r)$ as a polynomial of degree $r-1$ in terms of the form,

$$\pi_*(\pi_i)_*((v-ie)^a(w_1+e)^b(w_2-e^2)^c).$$

Expand each term, and reduce the powers of e using the basic relation,

$$e^3 + w_1 e^2 + w_2 e = 0.$$

Push out to F', and use the following three identities:

$$b_*[F_i] = [F \times X_i] \text{ and } b_* e = 0 \text{ and } b_* e^2 = -[\Delta].$$

Thus $u(D,r)$ can be expressed as a polynomial of degree r in the $y(a,b,c)$.

The actual polynomial expression can be found much more efficiently and written much more compactly as follows. As in Section 1, define polynomials $P_r(a_1, \ldots, a_r)$ by the formal identity in t,

$$\textstyle\sum_{r\ge 0} P_r t^r/r! = \exp\bigl(\sum_{q\ge 1} a_q t^q/q!\bigr).$$

This time, however, the variables a_q will not be replaced by numbers, but by classes on Y. These classes are certain linear combinations of the $y(a,b,c)$ of Display (4.5), and are computed as follows.

Given a formal monomial $(v-ie)^a(w_1+e)^b(w_2-e^2)^c$, expand it. Reduce the result modulo the relation $e^3 = -w_1e^2 - w_2 e$, collect the terms involving e^2, and denote the coefficient of e^2 by $q_i(a,b,c)$. Set

$$Qx_i(v^a w_1^b w_2^c) = -q_i(a,b,c).$$

Extend the definition of $Qx_i(\bullet)$ by linearity to all polynomials in v, w_1, w_2.

Define polynomials b_q in v, w_1, w_2 recursively by setting $b_1 := [X_2]$ and

$$b_{q+1} := P_q(Qx_2(b_1), \ldots, Qx_2(b_q))[X_2] - 3!\tbinom{q}{3}P_{q-3}(Qx_3(b_1), \ldots, Qx_3(b_{q-3}))[X_3] \\ + 3281 \cdot 7!\tbinom{q}{7}P_{q-7}(Qx_4(b_1), \ldots, Qx_4(b_{q-7}))[X_4],$$

where the $[X_i]$ are the polynomials in v, w_1, w_2 introduced early in the section. Finally, take v, w_1 and w_2 to be the Chern classes defined in Display (4.1), and set

$$a_q := \pi_*(b_q) \text{ for } q = 1, \ldots, r.$$

Then it can be checked by brute force that

$$u(D, r) = P_r(a_1 \ldots, a_r)/r!\,.$$

It would be good to have a conceptual proof that our two mechanical procedures always produce the same polynomials.

We are now ready to prove Theorems (1.1) and (1.2). So take $F = S \times Y$ where S is the given surface and where Y is the n-dimensional parameter projective space of the given complete linear system $|\mathcal{L}|$ on S. Take $D \subset F$ to be the total space of $|\mathcal{L}|$. Then $\mathcal{O}_F(D)$ is equal to $\mathcal{L} \otimes \mathcal{O}_Y(1)$, and so

$$v = l + h \text{ where } l := c_1(\mathcal{L}) \text{ and } h := c_1(\mathcal{O}_Y(1)).$$

Also, $\Omega^1_{F/Y}$ is simply the pullback of Ω^1_S. Hence, only four of the classes $y(a, b, c)$ are nonzero, namely,

$$y(r+2, 0, 0),\ y(r+1, 1, 0),\ y(r, 2, 0),\ y(r, 0, 1),$$

and their values are, respectively,

$$\tbinom{r+2}{2}dh^r,\ (r+1)kh^r,\ sh^r,\ xh^r,$$

where d, k, s, and x are the four numbers defined at the beginning of Section 1.

Using the procedure of the preceding paragraph, we get

$$u(D, r) = (P_r(d, k, s, x)/r!)h^r$$

where P_r is viewed as a polynomial in d, k, s, x as in Theorem (1.1). Finally, N_r is just the degree of $u(D, r)$ because of Proposition (4.2) and of the following little lemma; apply the lemma with $Y(rA_1)$ as U. Thus Theorem (1.1) is proved.

Lemma (4.7) *In the above setup, let U be a reduced subscheme of Y of pure codimension r, and U' its boundary. Assume that U parameterizes curves without multiple components, and set $m := n - r$. Then there exists a nonempty open subset S_m of $S^{\times m}$ with this property: let M be the linear r-space representing those curves that pass through the points of any given m-tuple in S_m; then $M \cap U$ is finite and reduced, and $M \cap U'$ is empty.*

Indeed, in D, form the smooth locus D_0 of the projection $D \to Y$. Form the fibered product $D_0^{\times m}$. Then $D_0^{\times m} \to Y$ is smooth. Hence $D_0^{\times m} \times_Y U$ is reduced. Moreover, it is dense in $D^{\times m} \times_Y U$. Also, $D^{\times m} \times_Y U$ is of pure dimension $2m$. Consider the natural map,

$$D^{\times m} \times_Y U \to S^{\times m}.$$

By Sard's theorem, its fibers are finite and reduced over a nonempty open subset S_m of $S^{\times m}$. These fibers are simply the $M \cap U$.

On the other hand, $D^{\times m} \times_Y U'$ is of dimension at most $2m-1$. So the map,

$$D^{\times m} \times_Y U' \to S^{\times m},$$

cannot be surjective. Hence, its fibers are empty over a nonempty subset of S_m. Replace S_m by this subset. Then these fibers are the $M \cap U'$. Thus the lemma is proved.

To prove Theorem (1.2), we work with the following seven classes on Y,

$$[U(D_4+(r-4)A_1)] \text{ for } r=4,5,6,7,\ [U(D_6)],\ [U(D_6+A_1)],\ [U(E_7)].$$

Again, Proposition (4.2) and Lemma (4.7) imply that the degrees of these classes are just the numbers we seek.

From the discussion in the second paragraph after (4.6), we get the equality,

$$[U(D_4+(r-4)A_1)] = u(D_3, r-4).$$

That discussion is set-theoretic, but the corresponding cycles are reduced because the requisite genericity hypothesis holds; it holds initially because of Proposition (4.2), and subsequently because genericity propagates. Now, the class $u(D_3, r-4)$ was computed implicitly above as part of the recursion that led to formulas in Theorem (1.1). By making it explicit for $r = 4,5,6,7$, we obtain the first four formulas in Theorem (1.2).

Looking back, we see that we've established an equation of the form,

$$\textstyle\sum_{r\geq 0} u(D_3, r)t^r/r! = \exp\bigl(\sum_{q\geq 1} a_q t^q/q!\bigr),$$

at least modulo t^4, where the a_q are linear combinations of certain classes on X_3, namely, of the analogues for F_3/X_3 and D_3 of the $y(a,b,c)$ of Display (4.5).

We obtain the last three formulas in Theorem (1.2) from the following three formulas, which we'll prove in a moment:

$$[U(D_6)] = \pi_*[U_2(D_4+A_1)] - 2[U(D_4+2A_1],\tag{4.8.1}$$

$$[U(D_6+A_1)] = \pi_*[U_2(D_4+2A_1)] - 3[U(D_4+3A_1],\tag{4.8.2}$$

$$[U(E_7)] = \pi_*[U_2(D_6)] - [U(D_6+A_1)],\tag{4.8.3}$$

where $U_2(\mathbf{D})$ is the analogue of $U(\mathbf{D})$ for F_2/X_2 and D_2. We just discussed how to find the class $[U(D_4+jA_1)]$ for $j=0,1,2,3$. By applying that result with F/X and D replaced by F_2/X_2 and D_2, we obtain an expression for $[U_2(D_4+jA_1)]$. Then the first two formulas above yields expressions for $[U(D_6)]$ and $[U(D_6+A_1)]$. Applying the first of these with F/X and D replaced by F_2/X_2 and D_2, we obtain an expression for $[U_2(D_6)]$. Then the third formula above yields an expression for $[U(E_7)]$.

It remains to prove Formulas (4.8.1), (4.8.2) and (4.8.3). Let C be a curve (or fiber) in the family D_2/X_2 with diagram $D_4+(j+1)A_1$ where $0 \leq j \leq 1$. If C does not contain the exceptional curve E in its ambient fiber of F_2/X_2, then C is the proper transform of a curve that is in the family D/Y and that has, at the blowup center, a singularity with diagram A_k where $1 \leq k \leq 3$. Therefore, the full diagram of this curve is one of the following:

$$D_4+(j+2)A_1,\ D_4+(j+1)A_1+A_2,\ \text{or } D_4+jA_1+A_3.$$

On the other hand, if C contains E, then since C is reduced, $C = C' + E$ where C' is the proper transform of a curve that has, at the blowup center, a singularity with diagram D_6.

Thus $X_2(D_4 + (j+1)A_1)$ is carried by π into the union of the sets $Y(\mathbf{D})$ where $\mathbf{D}$ is one of the following four diagrams:

$$D_4 + (j+2)A_1,\ D_4 + (j+1)A_1 + A_2,\ D_4 + jA_1 + A_3 \text{ or } D_6 + jA_1.$$

In fact, $X_2(D_4 + (j+1)A_1)$ is carried onto the union because the preceding analysis is reversible. Of these $\mathbf{D}$, only $D_4 + (j+2)A_1$ and $D_6 + jA_1$ have $\operatorname{cod}(\mathbf{D}) = 6 + j$; the other two $\mathbf{D}$ have $\operatorname{cod}(\mathbf{D}) \geq 7+j$. Moreover, above a point in $Y(D_4+(j+2)A_1)$ there are $j+2$ points in $X_2(D_4+(j+1)A_1)$, whereas above a point in $Y(D_6+jA_1)$, there is only one point. It now follows from Proposition (4.2) that Formulas (4.8.1) and (4.8.2) hold.

Formula (4.8.3) may be proved similarly. Let C be a curve in the family D_2/X_2 with diagram D_6. If C does not contain E, then C is the proper transform of a curve that is in D/Y and that has, at the blowup center, a singularity with diagram A_k where $1 \leq k \leq 3$. If C contains E, then $C = C' + E$ where C' is the proper transform of a curve that has, at the blowup center, a singularity with diagram E_7. In fact, this analysis is reversible. Thus $X_2(D_6)$ is carried by π onto the union of the sets $Y(\mathbf{D})$ where $\mathbf{D}$ is one of the following three diagrams:

$$D_6 + A_1,\ D_6 + A_2, \text{ or } E_7.$$

Now, $\operatorname{cod}(D_6 + A_1) = 7$ and $\operatorname{cod}(E_7) = 7$, but $\operatorname{cod}(D_6 + A_2) = 8$. Moreover, above a point in either $Y(D_6 + A_1)$ and $Y(E_7)$, there is only one point in $X_2(D_6)$. It now follows from Proposition (4.2) that the final formula (4.8.3) holds.

Thus both Theorems (1.1) and (1.2) are proved!

5. Vainsencher's treatment

Fix $r \leq 7$. Vainsencher [**28**] enumerated the r-nodal curves in a "suitably general" linear system of dimension r on a smooth, irreducible, projective surface S. By "suitably general," he meant that the system is a subsystem of a complete system $|\mathcal{L}|$, and as such corresponds to a point in a suitable nonemtpy open subset of the Grassmannian of subsystems; in addition, $\mathcal{L} = \mathcal{M}^{\otimes m}$ where $\mathcal{M}$ is ample and $m \geq m_0$ for a suitable m_0. He established the existence of the open subset and of m_0 if $r \leq 6$. If $r = 7$, he said in his Section 7 that then his argument does not apply. Moreover, he said that his final formula does not yield integers in certain examples; however, he later found a computational error [pvt. comm., Dec. 1997].

In this section, we relate Vainsencher's enumeration to our own. Thereby, we justify his approach on the basis of ours for every r, including $r = 7$. Furthermore, we prove that the term "suitably general" may be used in the following more precise sense, which was specified in Theorem (1.1) of the introduction: the system is *suitably general* if it is a subsystem defined by the condition to pass through $n - r$ general points where n is the dimension of $|\mathcal{L}|$, and if $\mathcal{L} = \mathcal{M}^{\otimes m} \otimes \mathcal{N}$ where $\mathcal{N}$ is spanned, $\mathcal{M}$ is very ample, and $m \geq 3r$. Of course, it follows formally from our result that there exists a nonemtpy open subset of the Grassmannian of subsystems of $|\mathcal{L}|$, whose points represent systems that work.

Vainsencher used an ad hoc recursive definition of the type of a singularity of a curve C on S. Namely, he called a sequence of points $(x_1, x_2, \ldots, x_r)$ a "singularity

of type" $(m_1, m_2, \dots, m_r)$ of C if x_1 is a point of C of multiplicity at least m_1 and if, on the blowup of S at x_1, the sequence $(x_2, \dots, x_r)$ is a singularity of type $(m_2, \dots, m_r)$ on the difference $C_1 - m_1E_1$, where C_1 is the total transform of C, and E_1 is the exceptional divisor. To avoid confusion with the traditional notion of singularity, call $(x_1, x_2, \dots, x_r)$ a *singularity sequence*.

Vainsencher said that $(x_1, x_2, \dots, x_r)$ is of *strict type* if x_1 is a point of C of multiplicity exactly m_1, if x_2 lies off of E_1, and if $(x_2, \dots, x_r)$ is of strict type on $C_1 - m_1E_1$; otherwise, he said that the sequence is of *weak type*. He wrote $m^{[s]}$ in place of a string of s repetitions of m; for example, $(3, 2^{[3]})$ stands for $(3, 2, 2, 2)$.

Fix a sequence $(x_1, \dots, x_r)$ of type $(2^{[r]})$. Distinguish the x_i that actually lie in C, and at their union, consider the singularity of C. Let $\mathbf{D}$ be its minimal Enriques diagram, and set $k := \operatorname{cod}(\mathbf{D})$. We are going to prove that $k \geq r$, and to determine when $k = r$. To do so, assume that $k \leq r$; we are going to analyze the cases, and to prove that $k = r$ or to find a contradiction. To begin, assume in addition that only one of the x_i lies in C; necessarily, it is x_1.

Suppose $x_1 \in C$ is a singularity of type A_k. Then x_1 is resolved by a sequence of either $k/2$ or $(k+1)/2$ blowups at centers that are points of multiplicity 2. Hence $(k+1)/2 \geq r$. However, $k \leq r$. Hence $k = r = 1$. Thus x_1 is of type A_1, and (x_1) is of strict type (2).

Suppose $x_1 \in C$ is of type D_k or E_k; these are the only other types of singularities with $k \leq 7$. Then x_1 is of multiplicity 3. So $C_1 - 2E_1$ is equal to $C' + E_1$ where C' is the strict transform of C. We now prove that x_1 must be of type D_4, D_6, or E_7.

Suppose x_1 is of type D_k. Then $C' + E_1$ has, along E_1, a singularity of type $3A_1$ if $k = 4$, of type $A_1 + A_3$ if $k = 5$, of type $A_1 + D_4$ if $k = 6$, or of type $A_1 + D_5$ if $k = 7$. Hence, if $k = 4$, then $r \leq 4$, and so $k = r$ since $k \leq r$. Moreover, it follows that, if $k = 6$, then $r \leq 6$, and so $k = r$. Now, an A_3 singularity is resolved by two blowups at centers that are points of multiplicity 2. Hence, if $k = 5$, then $r \leq 4$, a contradiction. Moreover, it follows that, if $k = 7$, then $r \leq 6$, a contradiction.

Suppose x_1 is of type E_k with $k = 6$ or $k = 7$. Then $C' + E_1$ has a singularity at x_2 of type A_5 if $k = 6$, or of type D_6 if $k = 7$; moreover, $x_3, \dots, x_r$ lie infinitely near x_2. Hence, it follows from the analysis above of types A_5 and D_6 that $k = r = 7$.

Suppose now that several of the x_i lie in C. For each x_i in C, let $\mathbf{D}_i$ be the diagram of the singularity of C at x_i, set $k_i = \operatorname{cod}(\mathbf{D}_i)$, and denote by r_i the number of x_j that lie infinitely near x_i, including x_i. Apply the above analysis to the sequence of these x_j. It yields $k_i \geq r_i$, and enumerates the possible types when $k_i = r_i$. Now, $\sum k_i = k$ and $\sum r_i = r$. Hence, $k \geq r$, and if $k = r$, then $k_i = r_i$. Putting it all together, we obtain the second column of Table 5-1, which lists the possible types of singularities when $k = r$.

Table 5-1
Sequences of type $(2^{[r]})$ with $\operatorname{cod}(\mathbf{D}) = r$

r	type of singularity	# of permutations
1, 2, 3	rA_1	$r!$
4	$4A_1,\ D_4$	$4!,\ 6$
5	$5A_1,\ D_4 + A_1$	$5!,\ 30$
6	$6A_1,\ D_4 + 2A_1,\ D_6$	$6!,\ 180,\ 30$
7	$7A_1,\ D_4 + 3A_1,\ D_6 + A_1,\ E_7$	$7!,\ 1260,\ 210,\ 30$

Given a singularity of a type listed in Table 5-1, there is a sequence $(x_1, \dots, x_r)$ of type $(2^{[r]})$ giving rise to it. Indeed, a second look at the analysis above shows the existence of a sequence. The set of points is uniquely determined, but their order is subject to permutation. The number of permutations is not hard to find; it is listed, case by case, in the third column of the table. (In [**20**], Liu provided a combinatorial device, "admissible graphs," to help find them.) Two examples are worked out next to illustrate how to find these numbers directly.

For example, a D_4 singularity $x_1 \in C$ arises from the sequence $(x_1, \dots, x_4)$ where x_2, x_3, and x_4 are the three points infinitely near x_1. The three points are subject to 6 permutations. Similarly, a $D_4 + A_1$ singularity arises from the sequence $(x_1, \dots, x_5)$ where $x_1, \dots, x_4$ are as before and x_5 is a node. The four points $x_2, \dots, x_5$ are nodes on the blowup of S at x_1; so the four are subject to 24 permutations. In addition, the five points $x_1, \dots, x_5$ may be cyclically permuted to $x_5, x_1, \dots, x_4$, and then the three points x_2, x_3, x_4 are subject to 6 permutations. In total, the sequence $(x_1, \dots, x_5)$ is subject to 30 permutations.

Let C vary now in a linear system that is suitably general in the sense specified above in the second paragraph of the section. Then, by Section 4, only finitely many of these C have a diagram $\mathbf{D}$ with $\operatorname{cod}(\mathbf{D}) = r$, and no C has $\operatorname{cod}(\mathbf{D}) > r$. Hence there is a finite number of singularity sequences of type $(2^{[r]})$ on all the C; denote this number by $N(2^{[r]})$. Therefore, with the notation of Section 1, we find the following formulas:

$$
\begin{aligned}
&N(2^{[r]}) = r!\, N_r \text{ for } r = 1, 2, 3,\\
&N(2^{[4]}) = 4!\, N_4 + 6\, N(3),\\
&N(2^{[5]}) = 5!\, N_5 + 30\, N(3, 2),\\
&N(2^{[6]}) = 6!\, N_6 + 180\, N(3, 2, 2) + 30\, N(3(2)),\\
&N(2^{[7]}) = 7!\, N_7 + 1260\, N(3, 2, 2, 2) + 210\, N(3(2), 2) + 30\, N(3(2)').
\end{aligned}
\tag{5.1}
$$

Vainsencher found similar formulas; see Proposition 4.1 and Section 7 in [**28**]. However, instead of the coefficients 180 and 1260, he had 90 and 210 (or literally, 1260/6), since he enumerated, not the curves with singularities of types $D_4 + 2A_1$ and $D_4 + 3A_1$, but the sequences of strict types $(3, 2, 2)$ and $(3, 2, 2, 2)$, which are subject to 2! and 3! permuations respectively.

Consider the set of all singularity sequences of type $(2^{[r]})$. It underlies a natural scheme, which is formed, using the setup of Section 4, as follows. If $r = 1$, take the scheme X_2. If $r = 2$, take the corresponding scheme of the family D_2/X_2. If $r \geq 3$, repeat this procedure $r - 2$ more times. Now, using the methods of Section 4, it is not hard to find an expression in terms of the four numbers d, k, s, x of Section 1 for the degree of this scheme, and to check that this expression is equal to the polynomial obtained from the above formulas and from Theorems (1.1) and (1.2). It follows in particular that the scheme is reduced, and that its degree (when considered as a cycle on Y) is equal to $N(2^{[r]})$; the reducedness also follows directly from Section 4.

Vainsencher obtained his formulas for the N_r from his version of the formulas in Display (5.1) and his version of Theorem (1.2). Thus we have indeed justified and extended his approach.

The formulas in Display (5.1) are implicit in our proof of Theorem (1.1) in Section 4, and it is interesting to make them explicit as we are about to do. In

particular, we recover the coefficients using less combinatorics. Furthermore, as in our proof of Theorem (1.1), we must work in a more general setup: we must let C vary in an algebraic system on a family of surfaces satisfying the genericity hypothesis that the assertions in Parts (1) and (2) of Proposition (4.2) both hold. Thus, in this more general setup, we now establish the corresponding versions of the formulas in Display (5.1).

First of all, if $r \le 4$, then the formula for $N(2^{[r]})$ results immediately from the recursive formula (4.6) by taking degrees.

To obtain the formula for $N(2^{[5]})$ in Display (5.1), apply the formula for $N(2^{[4]})$ to D_2/X_2, getting

$$N(D_2; 2^{[4]}) = 4!\, N_4(D_2) + 6\, N(D_2; 3).$$

Identifying the terms, we find

$$N(2^{[5]}) = 4! \deg \pi_* u(D_2, 4) + 6\, N(3, 2).$$

Hence (4.6) yields

$$N(2^{[5]}) = \big(5!\, N_5 + 24\, N(3,2)\big) + 6\, N(3,2) = 5!\, N_5 + 30\, N(3,2),$$

which is the formula in Display (5.1).

Similarly, to obtain the formula for $N(2^{[6]})$, apply the formula for $N(2^{[5]})$ to D_2/X_2, getting

$$N(D_2; 2^{[5]}) = 5!\, N_5(D_2) + 30\, N(D_2; 3, 2).$$

Now, $N(D_2; 3, 2) = 2\, N(3, 2, 2) + N(3(2))$ follows from Formula (4.8.1) by taking degrees. Identifying the several terms, we find

$$N(2^{[6]}) = 5!\, \deg \pi_* u(D_2, 5) + 60\, N(3, 2, 2) + 30\, N(3(2)).$$

Hence (4.6) yields

$$N(2^{[6]}) = \big(6!\, N_6 + 120\, N(3, 2, 2)\big) + 60\, N(3, 2, 2) + 30\, N(3(2)).$$

Combining the second and third terms yields the formula in Display (5.1).

Finally, to obtain the formula for $N(2^{[7]})$, apply the formula for $N(2^{[6]})$ to D_2/X_2, and identify the terms, getting similarly

$$\begin{aligned} N(2^{[7]}) = \big(7!\, N_7 + 720\, N(3, 2, 2, 2)\big) &+ 180\big(3\, N(3, 2, 2, 2) + N(3(2), 2)\big) \\ &+ 30\big(N(3(2), 2) + N(3(2)')\big). \end{aligned}$$

Combining terms yields the formula in Display (5.1), and completes our work.

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